

Mathematics of Harmony

Piano Keyboard

88 notes forming a chromatic scale, each step being a “semitone” half-step or minor 2nd.

Twelve semitones form one octave.

Key is a subset of 7 notes in an octave. White notes are the key of C.

Notes labeled with number, different numbering schemes, this one is MIDI.

MIDI number	Note name	Keyboard	Frequency Hz
21	A0		27.500
22	B0		30.868
23	C1		32.703
24	D1		36.708
25	E1		38.891
26	F1		43.654
27	G1		48.999
28	A1		51.913
29	B1		58.270
30	C2		65.406
31	D2		73.416
32	E2		82.407
33	F2		87.307
34	G2		97.999
35	A2		110.000
36	B2		123.470
37	C3		130.810
38	D3		146.830
39	E3		164.810
40	F3		174.610
41	G3		196.000
42	A3		220.000
43	B3		246.940
44	C4		261.630
45	D4		293.670
46	E4		329.630
47	F4		349.230
48	G4		392.000
49	A4		440.000
50	B4		493.880
51	C5		523.250
52	D5		587.330
53	E5		659.260
54	F5		698.460
55	G5		783.990
56	A5		880.000
57	B5		987.770
58	C6		1046.500
59	D6		1174.700
60	E6		1318.500
61	F6		1396.900
62	G6		1568.000
63	A6		1760.000
64	B6		1975.500
65	C7		2093.000
66	D7		2349.300
67	E7		2637.000
68	F7		2793.000
69	G7		3136.000
70	A7		3520.000
71	B7		3951.100
72	C8		4186.000

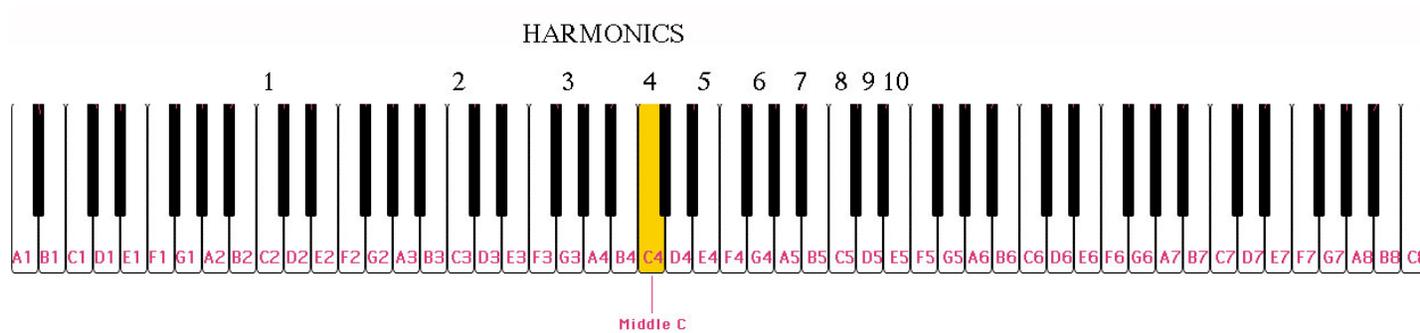
J. Wolfe, UNSW

Intervals

Represent a certain number of semitones, e.g., perfect 5th is seven semitones.

Named for distance along the major scale, e.g., major 3rd or minor (flatted) 6th.

Harmonic Series



Basis of the Pythagorean Scale (fretted instruments, ancient Greece).

Many intervals are perfectly in tune (whole number ratios), e.g., octave, perfect 5th, major 3rd, dominant 7th, etc.

However, not every 5th on the keyboard is perfect.

Only certain keys sound good.

Tempered Scale

Increase frequency by $\sqrt[12]{2}$ (an irrational number, 1.059...), or just less than 6% with each semitone.

One semitone is exactly 100 “cents.”

After 12 steps you have a net increase of a factor of 2, one octave.

Could now compose in all keys (1500s, Bach’s Well Tempered Clavier).

Irrational numbers cannot be represented by ratios of integers, so intervals now must contain “beats.”

Stretching

Inharmonicity – Thickness of string causes higher harmonics to go sharp.

Thus, octaves (or higher harmonic) must be tuned sharp to sound right. No perfect compromise.

Musical Language of Harmony

Major Scale

Scales are sets of notes (generally 5 to 12 notes) with a root.

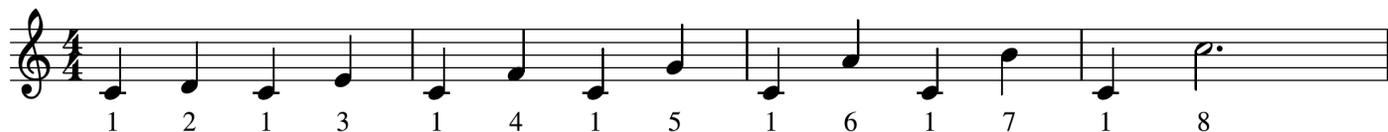
Certain series of whole steps and half steps

Intervals (up as well as down) named by distance along the major scale

<u>half steps</u>	<u>interval</u>
0	unison
1	minor or flatted 2 nd , half-step
2	2 nd , major 2 nd , whole-step
3	minor or flatted 3 rd
4	3 rd , major 3 rd
5	4 th or perfect 4 th
6	augmented 4 th , flatted (diminished) 5 th , tritone
7	5 th or perfect 5 th
8	augmented 5 th or flatted 6 th
9	6 th or major 6 th
10	flatted 7 th , dominant 7 th , or just 7 th (even though it's flatted!)
11	major 7 th (not generally called just 7 th)
12	octave
13	flatted 9 th
14	9 th

Intervals have inversions (e.g. a 5th is an inverted 4th , a major 3rd is inverted minor 6th)

Ear training exercise to learn ascending intervals.



Triads (combinations of major and minor thirds)

Chords are sets of notes (generally 3 to 6 notes) with a root (*different than root of scale*)

Major

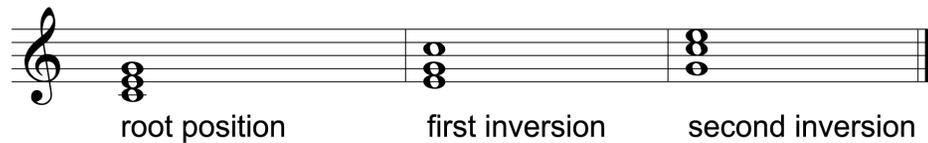
Minor

Diminished

Augmented

Inversions

Flipping the notes so that the root is not on the bottom.



Hearing the root in an inversion is a basic ear-training skill. Sing the chord in root position.

Major Modes

Ionic Mode (root at I)

Other Modes of the major (diatonic) scale (roots other than I)

IV (Lydian) and V (Mixolydian) are major scales with only one altered note. (#4 and *b*7)

II (Dorian) and VI (Aeolian – “relative minor”) have the first 5 notes like major but with a minor 3rd, varying in the 6ths and 7ths.

III (Phrygian) and VII (Locrian) are weirder, with even the first five notes altered.

Major Modes are made of a particular sequence of major, minor, and diminished triads.

Thus, Ionic mode is this sequence of triads:



1 ma-jor 2 mi-nor 3 mi-nor 4 ma-jor 5 ma-jor 6 mi-nor 7 dimin-ished
1 3 5 2 4 6 3 5 7 4 6 8 5 7 9 6 8 10 7 9 11

Major Modes are also completely defined by I, IV, and V chords.

Dorian mode has a minor 1 chord, a major 4 chord, and a minor 5 chord,

Aeolian mode has a minor 1 chord, a *minor* 4 chord, and a minor 5 chord.

4-note chords in major scale

Major 7th

Dominant 7th

Minor 7th

Half-Diminished 7th (minor 7 flat 5), or inverted minor 6th chord

Substitution chords

C major shares two notes with E minor.

F major shares two notes with D minor.

C7 shares two notes with F#7 (tritone substitution).

Chords with ambiguous roots (symmetry)

Diminished 7th

Actually, only three unique ones.

Beethoven

Augmented triad

Actually, only four unique ones.

Debussy

Voicings

Many other permutations than just the contiguous notes of the standard inversions.

Circle of Fifths

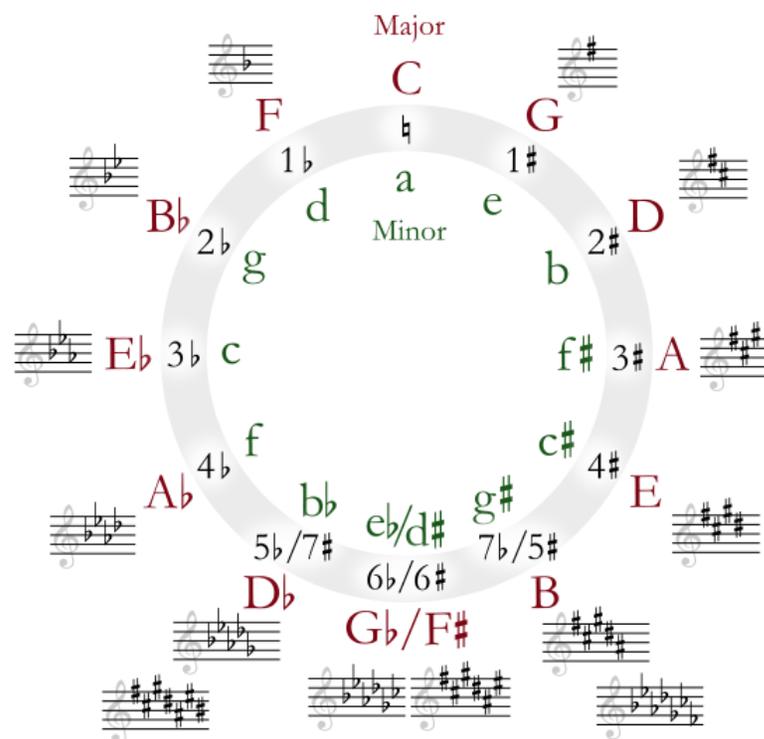
Major scale grouped on one side.

Black notes grouped on the other.

Pentatonic scale is any contiguous group of five, universal scale across cultures.

Tritone – flatted fifth – straight across, (Devil’s interval, blues).

Keys identified by number of sharps or flats.



Circle of Fifths Excercise Example

Jaki Byard

C F B^b E^b A^b D^b

7 F[#] B E A D G

13 C G D A E B

19 F[#] D^b A^b E^b B^b F repeat ad infinitum

transcribed by
George Stetten,
former student of
Jaki Byard



Jaki Byard was a founding faculty member of the Contemporary Improvisation department at the New England Conservatory.

Other Scales

Chromatic

All 12 notes, all half-steps.

Whole tone

Because of symmetry, actually only 2 different whole-note scales.

Each encompasses 3 augmented triads

Diminished or “Octatonic” (8 notes)

Alternating whole-note and half-note steps, because of symmetry, actually only 3 different whole-note scales.

Each encompasses 2 diminished triads

Minor scales with other variations flatted 6ths and 7ths with the first 5 notes normal.

Aeolian Mode (Natural Minor) – b6, b7

Dorian Mode – 6 b7

Melodic 6 7 – (up only, Natural on the way down!)

Harmonic – (b6 7)

Hungarian – (harmonic with #4)

Other Chords

Lots of permutations

Jazz chords often do not include root or fifth

e.g. tri-tone substitution

Chords over roots

Em/c = C maj7 (lower case letters for notes, upper case for triads)

C#dim7/c = C7b9

Poly-chords

G/C = Cmaj7,9

Chord Sequences

Resolutions 4-1 (amen...) and 5-1 (traditional ending) moving to the neighboring scale in the circle of fifths.

“5 of the 5”, $2 = 5^2$, $6 = 5^3$, 6-2-5-1 (and there was Grandma...)

“4 of the 4”, $7b = 4^2$

Going to the relative minor 6m through the 3 (3 is the 5 of the 6)

The same “altered” note, a raised 5th, in the 3 major cord, plays a different role in the 4 minor cord.

Licks

Sequences of notes that are not thought of in terms of scale

Just in the fingers

Not easily transposed (except on a guitar/violin)

Learn to sing melodies that you play (George Benson, Masquerade, Breezin 1976)

(<https://www.youtube.com/watch?v=g-ibK5L2a4I>)