University of Pittsburgh
Department of Bioengineering

BIOE 1330
Biomedical Imaging
Homework #2 Solns

1. Concerning Moire patterns:
   a. What are they?
   b. What causes them?
   c. How can they be removed?
   d. When designing a system, how can you avoid introducing them into an image?

   a. Moire patterns are an artificial pattern that appears within an image sampled below the Nyquist frequency.

   b. They occur when image frequency content exceeds the sampling rate.

   c. Once they occur, they can't be removed, therefore it's important to keep them out.

   d. 1. Sampling rate could be increased so that it exceeds the Nyquist rate for the image, or 2. The frequency content of the image can be reduced using low pass filtering (blurring) to reduce it below half the desired sampling frequency.

2. Complex conjugates can be used to isolate real component of a complex signal. Adding the complex conjugate results in the removal of the imaginary portion, though the result must be divided by two, to maintain magnitude as:

   cos ω = $(e^{j\omega t} + e^{-j\omega t})/2$, e^(jwt) =coswt + j sinwt

   Why does *subtracting* the complex conjugate leave only the imaginary portion?

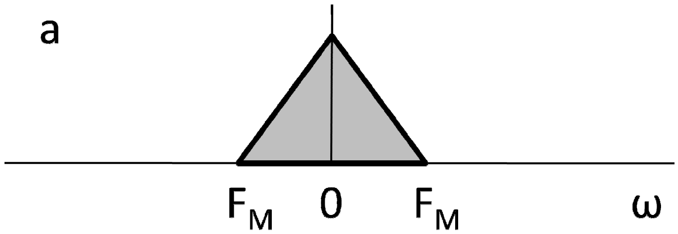
$$e^{j\omega t} - e^{-j\omega t} = (\cos\omega t + j\sin\omega t) - (\cos\omega t - j\sin\omega t)$$
$$j\sin\omega t + j\sin\omega t = 2j\sin\omega t$$

3. What's the difference between image noise, artifact, and distortion? Give an example of each. Which can be corrected, and which can't?

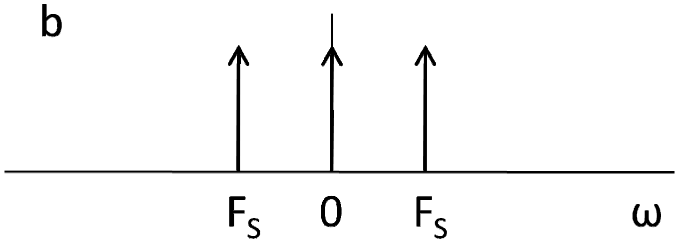
   Noise is image content not resulting from the patient, and devoid of a pattern (random), whereas artifact is unwanted content with a pattern or structure. Examples of noise, dust on a lens, static on an AM radio. Aliasing is an example of artifact. Distortion is scaling error or displacement in one or more directions. An image obtained through a fish-eye lens is distorted. If the characteristics of the distortion are known, distortions can be corrected to recover a "true" image. Noise and artifacts act as alterations of image content, and generally difficult or impossible to remove.

4. The frequency response of a signal is shown in figure a. Draw the frequency response after the signal has been sampled with the sampling train in figure b and explain if the signal can be fully reconstructed from the sampled for:

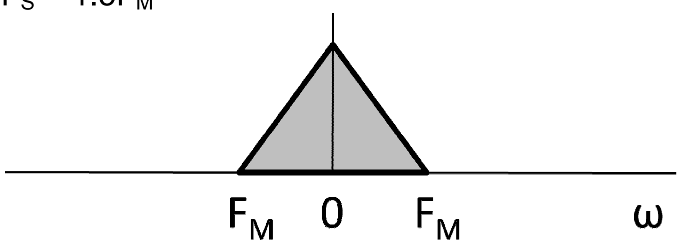
a

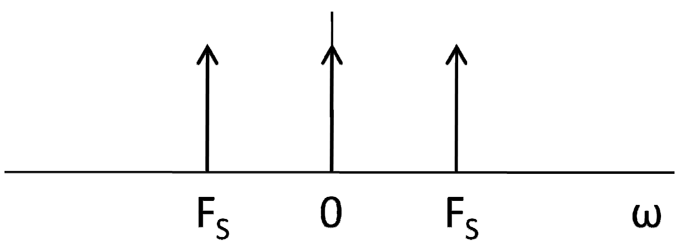


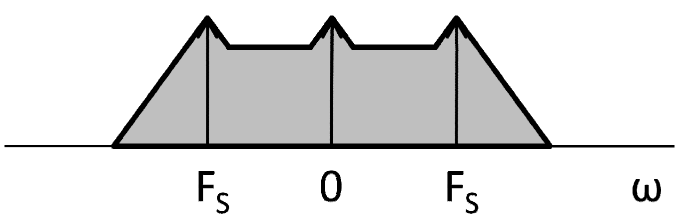
b



I. $F_S = 1.5F_M$

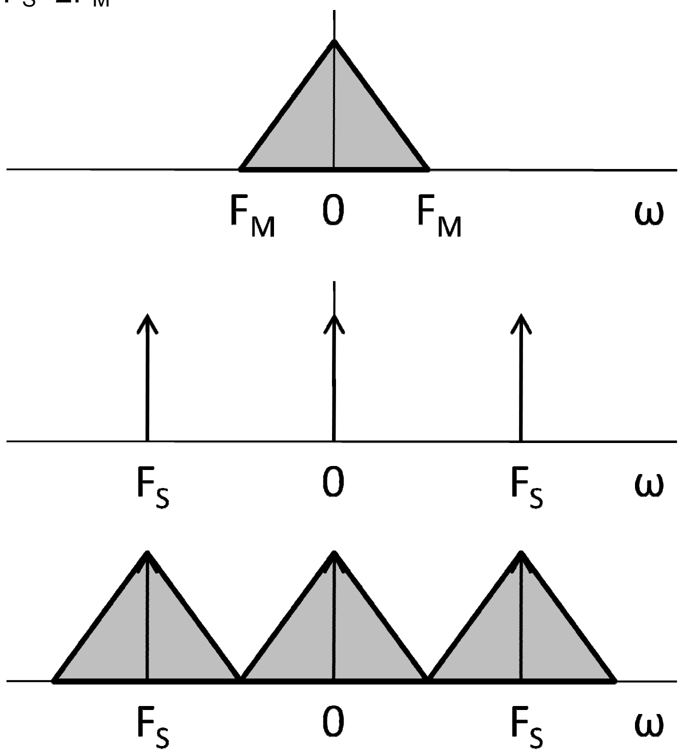






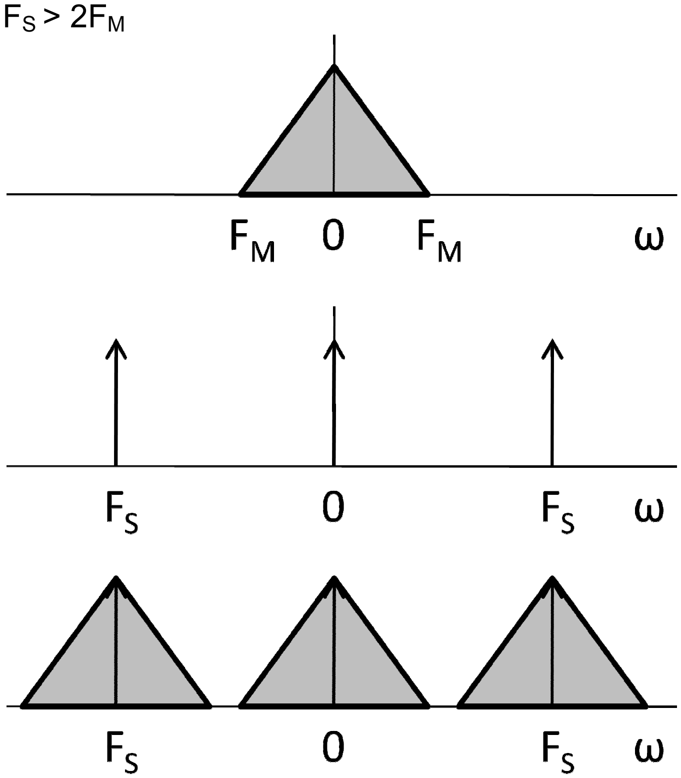
This sampled signal cannot be reconstructed due to aliasing

II. $F_S=2F_M$



This sampled signal can be reconstructed

III. $F_S > 2F_M$



This sampled signal can be reconstructed

5. Find the Fourier transform of the following continuous signals:
   I. $\delta(x,y)$

$$\mathcal{F}\big(\delta(x,y)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y) e^{-j2\pi(u0+v0)} dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y) dxdy = 1$$

   II. *f(x,y)* = (1/2)[sin(2*π*(*ax* + *by*)) + sin(2*π*(*ax* – *by*))]

$$\frac{1}{2}\big[\sin\big(2\pi(ax+by)\big) + \sin\big(2\pi(ax-by)\big)\big] = \sin 2\pi ax \cos 2\pi by$$

$$\mathcal{F}\ (\sin 2\pi ax \cos 2\pi by) = \mathcal{F}\ (\sin 2\pi ax)\mathcal{F}(\cos 2\pi by)$$

$$1. \mathcal{F}\ (\sin 2\pi ax) = \int_{-\infty}^{\infty} \sin 2\pi ax\, e^{-j2\pi(ux)} dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{j2\pi ax} - e^{-j2\pi ax}}{2j} e^{-j2\pi(ux)} dx = \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j2\pi x(u-a)} - e^{-j2\pi x(u+a)} dx$$

$$= \frac{1}{2j}[\delta(u-a) - \delta(u-a)]$$

$$2. \mathcal{F}(\cos 2\pi by)) = \int_{-\infty}^{\infty} \cos 2\pi by\, e^{-j2\pi(vy)} dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{j2\pi by} + e^{-j2\pi by}}{2} e^{-j2\pi(vy)} dy = \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi y(v-b)} + e^{-j2\pi y(v+b)} dy$$

$$= \frac{1}{2}[\delta(v-b) + \delta(v+b)]$$

$$\mathcal{F}(u,v) = \frac{1}{2j}[\delta(u-a) - \delta(u-a)]\frac{1}{2}[\delta(v-b) + \delta(v+b)]$$

$$= \frac{1}{4j}[\delta(u-a) - \delta(u-a)][\delta(v-b) + \delta(v+b)]$$

6. Prove equation 2.92 (The convolution property of the Fourier transform)

$$\mathcal{F}(f * g)(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)g(x-\xi,y-\eta)d\xi d\eta\right] e^{-j2\pi(ux+vy)} dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-\xi,y-\eta)e^{-j2\pi(ux+vy)} dxdy\right] d\xi d\eta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-\xi,y - \eta)\, e^{-j2\pi[u(x-\xi)+v(y-\eta)]} e^{-j2\pi(u\xi+v\eta)} dxdy\right] d\xi d\eta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-\xi,y - \eta)\, e^{-j2\pi[u(x-\xi)+v(y-\eta)]}\ dxdy\right] d\xi d\eta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(p,q)\, e^{-j2\pi[up+vq]}\, dpdq\right] d\xi d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\, \mathcal{F}(g)(u,v) d\xi d\eta$$
$$= \mathcal{F}(g)(u,v)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\, d\xi d\eta = \mathcal{F}(g)(u,v)\cdot\mathcal{F}(f)(u,v)$$
$$\mathcal{F}(g)(u,v)\cdot\mathcal{F}(f)(u,v) = \mathcal{F}(f * g)(u,v)$$

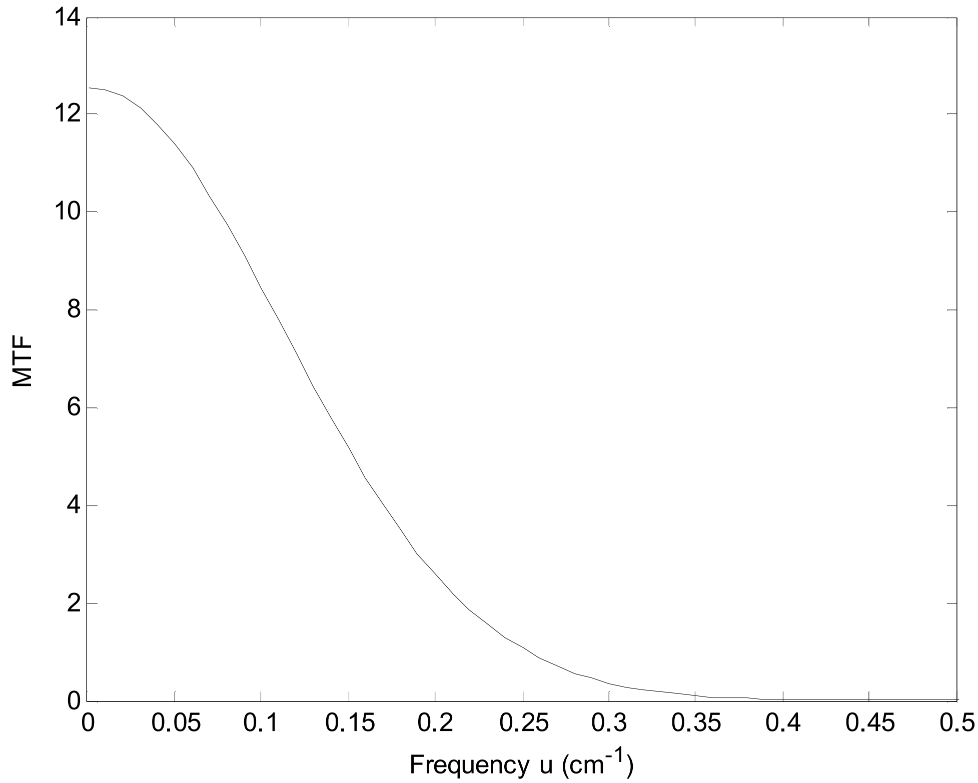
7. Consider an LSI system with a PSF given by *h*(*x*,*y*) = e^-(*x*^2 + *y*^2)/4
   I. Calculate the MTF associated with this system

$$MTF = |H(u,0)|$$
$$H(u,v) = \int_{-\infty}^{\infty} h(x,y)e^{-j2\pi(ux+vy)}dxdy$$
$$= \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}e^{-j2\pi(ux+vy)}dxdy$$

Using separability:

$$= \int_{-\infty}^{\infty} e^{-(x^2)/4}e^{-j2\pi(ux)}dx\int_{-\infty}^{\infty} e^{-(y^2)/4}e^{-j2\pi(vy)}dy$$
$$\int_{-\infty}^{\infty} e^{-(x^2)/4}e^{-j2\pi(ux)}dx = \int_{-\infty}^{\infty} e^{-(x^2+j8\pi ux)/4}dx$$
$$= \int_{-\infty}^{\infty} e^{-(x^2+j8\pi ux-16\pi^2u^2)/4}e^{-4\pi^2u^2}dx = e^{-4\pi^2u^2}\int_{-\infty}^{\infty} e^{-(x+j4\pi u)^2/4}dx$$
$$\sqrt{4\pi}e^{-4\pi^2u^2}$$
$$H(u,v) = 4\pi e^{-4\pi^2(u^2+v^2)}$$
$$MTF = 4\pi e^{-4\pi^2u^2}$$

II. Plot the MTF as a function of frequency



8. For a uniform distribution over the interval [a,b] prove

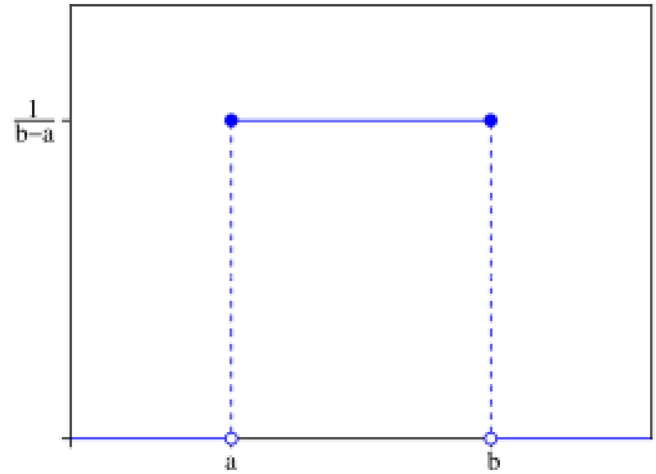
I. The pdf $p_N(\eta)$ is :

$1/(b - a)$ for $a \le \eta \le b$

0 otherwise

By definition: η is uniformly distributed in the interval $[a, b]$, $-\infty < a < b < \infty$, if

$$p_N(\eta) = \begin{cases} \frac{1}{b-a} & a \le \eta \le b \\ 0 & otherwise \end{cases}$$



II. The PDF $P_N(\eta)$ is:

0	for $a \le \eta$
$(\eta - a)/(b - a)$	for $a \le \eta \le b$
1	otherwise

$$P_N(\eta) = \int_{-\infty}^{\eta} \frac{1}{b-a} dn = \frac{1}{b-a}\int_{a}^{\eta} dn = \frac{\eta}{b-a} - \frac{a}{b-a} = \frac{\eta - a}{b-a}$$

III. The mean, $\mu_N = (a + b)/2$

$$\mu_N = \int_{-\infty}^{\infty} n p_N(n) dn = \int_{a}^{b} n \frac{1}{b-a} dn = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

IV. The variance, $\sigma_N^2 = (b - a)^2/12$

$$\sigma_N^2 = \int_{-\infty}^{\infty} (n - \mu_N)^2 p_N(n) dn = \int_{a}^{b} \left(n - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dn$$

$$\int_{a}^{b} \left(n - \frac{a+b}{2}\right)^2 \frac{1}{b-a} d\left(n - \frac{a+b}{2}\right) = \frac{1}{b-a}\int_{a-(a+b)/2}^{b-(a+b)/2} t^2\, dt$$

$$= \frac{1}{3(b-a)}\left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right] = \frac{(b-a)^2}{12}$$