Short Manifesto on Phasor Notation – George Stetten 2/11/07

To supplement the most recent lecture, I'd like to discuss the phasor notation $A\angle\theta$, which is widely used to mean $Ae^{j\theta}$. In its purest form, both O'Malley (at the bottom of p. 219 in the Schaum's Outline Series book) and Schertz (p. 261) agree that $A\angle\theta = Ae^{j\theta}$, exactly, as in the complex number defined by Euler's Identity,

$$Ae^{j\theta} = A\cos(\theta) + jA\sin(\theta)\,. \qquad (1)$$

The trouble is that when the $A\angle\theta$ notation is applied to real voltages and currents, it is not done in a standard manner and can be downright sloppy. O'Malley defines his choice of convention on page 221:

> "By definition, a *phasor* is a complex number associated with a phase-shifted sine wave such that, if the phasor is in polar form, its magnitude is the *effective* (*rms*) value of the voltage or current and its angle is the phase angle of the phase-shifted *sine* wave."

Thus when O'Malley says

$$A\angle\theta \text{ V}, \qquad (2)$$

he means the voltage function,

$$v(t) = \frac{A}{\sqrt{2}}\sin(\omega t + \theta). \qquad (3)$$

See examples 11.12 and 11.13 in O'Malley. The $\sqrt{2}$ derives from the fact that the *rms* (root-mean-square) magnitude of a sinusoid is $1/\sqrt{2}$ times the *peak* magnitude. This interpretation of $A\angle\theta$ is, however, by no means universal. As O'Malley himself admits,

> "There is not complete agreement on the definition of a phasor. Many electrical engineers use the sinusoidal *peak value* instead of the *effective [rms]* value. Also, they use the angle from the phase-shifted *cosine* wave instead of the *sine* wave."

To confuse matters further, when voltage or current is not involved, O'Malley still uses *sine* but reverts to *peak* rather than *rms*. (see example 11.14). Scherz uses "VAC" to mean *rms* voltage, and, as far as I can tell, leaves the *cosine* vs. *sine* issue completely ambiguous (He further compounds the confusion by erroneously leaving *j* out of two expressions of Euler's Identity in Figure 2.159).

Using *rms* has advantages when discussing AC power. Nonetheless, many prefer to define the A in $A\angle\theta$ to mean *peak* rather than *rms* voltage, and to base angle θ on *cosine* rather than *sine*. They would interpret $A\angle\theta$ V as

$$v(t) = A\cos(\omega t + \theta)\,. \qquad (4)$$

This strikes me as closer to the spirit of the full-blown correct algebraic expression derived from Euler's identity, which extracts the real part of the complex exponential as

$$A\cos(\omega t+\theta)=A\frac{e^{j(\omega t+\theta)}+e^{-j(\omega t+\theta)}}{2}. \tag{5}$$

Wikipedia agrees (http://en.wikipedia.org/wiki/Phasor_%28electronics%29) though they also mention the other conventions.

Engineers must deal with real voltages and currents, and to avoid having to write out Equation 5 for every current and voltage in a circuit, they like using the shorthand $A\angle\theta$. Because linear time-invariant systems can only scale the magnitude and shift the phase of a sinusoid, it doesn't usually matter if magnitude is *peak* or *rms*, or phase is relative to the *cosine* or the *sine*. Only changes in $A\angle\theta$ matter, and these will be the same for any of the various interpretations of the notation. Allow me to illustrate further.

Suspending for a moment the belief that voltages cannot be imaginary, and keeping in mind that, in the real world, a complex conjugate will always exist, let us use a single complex exponential $Ae^{j(\omega t+\theta)}=Ae^{j\theta}e^{j\omega t}$ to represent a complex voltage. At any given frequency ω, the only part you really need is $Ae^{j\theta}$, which incorporates the magnitude and phase of the total phasor (and *is*, in fact, the Fourier Transform at that frequency). That is what we mean, when we say, "$A\angle\theta$"; *it is the part of a given sinusoid that changes in a linear time-invariant system*.

Writing $Ae^{j\theta}$ instead of $A\angle\theta$ avoids ambiguity and permits traditional algebraic manipulation. This notation is generally what physicists use, as opposed to engineers. For example, to find the current in a capacitor generated by a voltage at frequency ω, let us use the complex number $\mathbf{V}=Ae^{j\omega t}$ to represent its magnitude and phase (O'mally uses bold for complex numbers, Schertz does not). By Ohm's law, the current is the voltage divided by the impedance, which for a capacitor is $1/j\omega C$. So,

$$\mathbf{I}=\frac{Ae^{j\omega t}}{1/j\omega C}=j\omega CAe^{j\omega t}=j\omega C\mathbf{V}=\omega Ce^{j\left(90^\circ\right)}\mathbf{V}. \tag{6}$$

since $j=e^{j\left(90^\circ\right)}$. That is to say, relative to the voltage, the current in a capacitor is shifted to the left by 90° and scaled by ωC (see Scherz, Fig. 2.106). In phasor notation, we could denote the real sinusoidal voltage and currents as

$$\mathbf{V}=A\angle\theta \tag{7}$$

$$\mathbf{I}=\frac{A\angle\theta}{\omega C\angle-90^\circ}=A\omega C\angle\left(\theta+90^\circ\right) \tag{8}$$

demonstrating again that current is shifted to the left by 90° and scaled by ωC. This will be true whether A means *rms* or *peak*, and whether θ is relative to *sine* or *cosine*. Thus the notation survives in spite of its ambiguity, and engineers use it widely to simplify complex systems, basically knowing what it means.