# Mathematical Tools for Physics

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# Complex Algebra

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When the idea of negative numbers was broached a couple of thousand years ago, they were considered suspect, in some sense not "real." Later, when probably one of the students of Pythagoras discovered that numbers such as $\sqrt{2}$ are irrational and cannot be written as a quotient of integers, legends have it that the discoverer suffered dire consequences. Now both negatives and irrationals are taken for granted as ordinary numbers of no special consequence. Why should $\sqrt{-1}$ be any different? Yet it was not until the middle 1800's that complex numbers were accepted as fully legitimate. Even then, it took the prestige of Gauss to persuade some. How can this be, because the general solution of a quadratic equation had been known for a long time? When it gave complex roots, the response was that those are meaningless and you can discard them.

### 3.1 Complex Numbers

As soon as you learn to solve a quadratic equation, you are confronted with complex numbers, but what is a complex number? If the answer involves $\sqrt{-1}$ then an appropriate response might be "What is *that*?" Yes, we can manipulate objects such as $-1+2i$ and get consistent results with them. We just have to follow certain rules, such as $i^2=-1$. But is that an answer to the question? You can go through the entire subject of complex algebra and even complex calculus without learning a better answer, but it's nice to have a more complete answer once, if then only to relax* and forget it.

An answer to this question is to define complex numbers as pairs of real numbers, $(a,b)$. These pairs are made subject to rules of addition and multiplication:

$$(a,b)+(c,d)=(a+c,b+d) \qquad \text{and} \qquad (a,b)(c,d)=(ac-bd,ad+bc)$$

An algebraic system has to have something called zero, so that it plus any number leaves that number alone. Here that role is taken by $(0,0)$

$$(0,0)+(a,b)=(a+0,b+0)=(a,b) \qquad \text{for all values of } (a,b)$$

What is the identity, the number such that it times any number leaves that number alone?

$$(1,0)(c,d)=(1\cdot c-0\cdot d,1\cdot d+0\cdot c)=(c,d)$$

so $(1,0)$ has this role. Finally, where does $\sqrt{-1}$ fit in?

$$(0,1)(0,1)=(0\cdot 0-1\cdot 1,0\cdot 1+1\cdot 0)=(-1,0)$$

and the sum $(-1,0)+(1,0)=(0,0)$ so $(0,1)$ is the representation of $i=\sqrt{-1}$, that is $i^2+1=0$. $\big[(0,1)^2+(1,0)=(0,0)\big]$.

This set of pairs of real numbers satisfies all the desired properties that you want for complex numbers, so having shown that it is possible to express complex numbers in a precise way, I'll feel free to ignore this more cumbersome notation and to use the more conventional representation with the symbol $i$:

$$(a,b) \;\longleftrightarrow\; a+ib$$

That complex number will in turn usually be represented by a single letter, such as $z=x+iy$.

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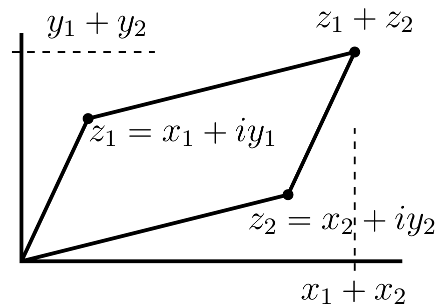
* If you think that this question is an easy one, you can read about some of the difficulties that the greatest mathematicians in history had with it: "An Imaginary Tale: The Story of $\sqrt{-1}$" by Paul J. Nahin. I recommend it.

The graphical interpretation of complex numbers is the Cartesian geometry of the plane. The $x$ and $y$ in $z = x + iy$ indicate a point in the plane, and the operations of addition and multiplication can be interpreted as operations in the plane. Addition of complex numbers is simple to interpret; it's nothing more than common vector addition where you think of the point as being a vector from the origin. It reproduces the parallelogram law of vector addition.




The *magnitude* of a complex number is defined in the same way that you define the magnitude of a vector in the plane. It is the distance to the origin using the Euclidean idea of distance.

$$|z| = |x + iy| = \sqrt{x^2 + y^2} \tag{3.1}$$

The multiplication of complex numbers doesn't have such a familiar interpretation in the language of vectors. (And why should it?)

### 3.2 Some Functions

For the algebra of complex numbers I'll start with some simple looking questions of the sort that you know how to handle with real numbers. If $z$ is a complex number, what are $z^2$ and $\sqrt{z}$? Use $x$ and $y$ for real numbers here.

$$z = x + iy, \qquad \text{so} \qquad z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

That was easy, what about the square root? A little more work:

$$\sqrt{z} = w \Longrightarrow z = w^2$$

If $z = x + iy$ and the unknown is $w = u + iv$ ($u$ and $v$ real) then

$$x + iy = u^2 - v^2 + 2iuv, \qquad \text{so} \qquad x = u^2 - v^2 \qquad \text{and} \qquad y = 2uv$$

These are two equations for the two unknowns $u$ and $v$, and the problem is now to solve them.

$$v = \frac{y}{2u}, \qquad \text{so} \qquad x = u^2 - \frac{y^2}{4u^2}, \qquad \text{or} \qquad u^4 - xu^2 - \frac{y^2}{4} = 0$$

This is a quadratic equation for $u^2$.

$$u^2 = \frac{x \pm \sqrt{x^2 + y^2}}{2}, \qquad \text{then} \qquad u = \pm\sqrt{\frac{x \pm \sqrt{x^2 + y^2}}{2}} \tag{3.2}$$

Use $v = y/2u$ and you have four roots with the four possible combinations of plus and minus signs. You're supposed to get only two square roots, so something isn't right yet; which of these four have to be thrown out? See problem 3.2.

What is the reciprocal of a complex number? You can treat it the same way as you did the square root: solve for it.

$$(x + iy)(u + iv) = 1, \qquad \text{so} \qquad xu - yv = 1, \qquad xv + yu = 0$$

Solve the two equations for $u$ and $v$. The result is

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2} \tag{3.3}$$

See problem 3.3. At least it's obvious that the dimensions are correct even before you verify the algebra. In both of these cases, the square root and the reciprocal, there is another way to do it, a much simpler way. That's the subject of the next section.

**Complex Exponentials**

A function that is central to the analysis of differential equations and to untold other mathematical ideas: the exponential, the familiar $e^x$. What is this function for complex values of the exponent?

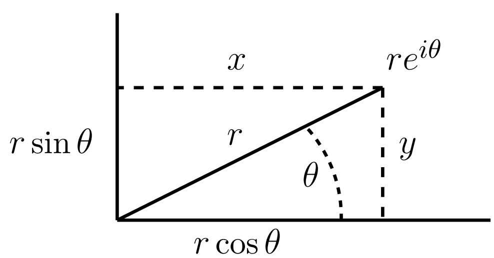
$$e^z = e^{x+iy} = e^x e^{iy} \tag{3.4}$$

This means that all that's necessary is to work out the value for the purely imaginary exponent, and the general case is then just a product. There are several ways to work this out, and I'll pick what is probably the simplest. Use the series expansions Eq. (2.4) for the exponential, the sine, and the cosine and apply it to this function.

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots \\ &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots + i\Big[y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\Big] = \cos y + i \sin y \end{aligned} \tag{3.5}$$

A few special cases of this are worth noting: $e^{i\pi/2} = i$, also $e^{i\pi} = -1$ and $e^{2i\pi} = 1$. In fact, $e^{2n\pi i} = 1$ so the exponential is a periodic function in the imaginary direction.

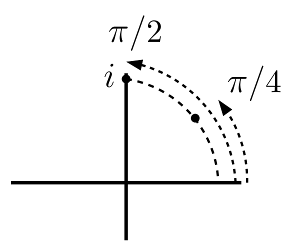
The magnitude or absolute value of a complex number $z = x + iy$ is $r = \sqrt{x^2 + y^2}$. Combine this with the complex exponential and you have another way to represent complex numbers.




$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{3.6}$$

This is the polar form of a complex number and $x + iy$ is the rectangular form of the same number. The magnitude is $|z| = r = \sqrt{x^2 + y^2}$. What is $\sqrt{i}$? Express it in polar form: $\big(e^{i\pi/2}\big)^{1/2}$, or better, $\big(e^{i(2n\pi + \pi/2)}\big)^{1/2}$. This is

$$e^{i(n\pi + \pi/4)} = \big(e^{i\pi}\big)^n e^{i\pi/4} = \pm(\cos\pi/4 + i\sin\pi/4) = \pm\frac{1+i}{\sqrt{2}}$$



### 3.3 Applications of Euler's Formula

When you are adding or subtracting complex numbers, the rectangular form is more convenient, but when you're multiplying or taking powers the polar form has advantages.

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} \tag{3.7}$$

Putting it into words, you multiply the magnitudes and add the angles in polar form.

From this you can immediately deduce some of the common trigonometric identities. Use Euler's formula in the preceding equation and write out the two sides.

$$r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2\big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big]$$

The factors $r_1$ and $r_2$ cancel. Now multiply the two binomials on the left and match the real and the imaginary parts to the corresponding terms on the right. The result is the pair of equations

$$\begin{aligned}\cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1+\theta_2) &= \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\end{aligned} \tag{3.8}$$

and you have a much simpler than usual derivation of these common identities. You can do similar manipulations for other trigonometric identities, and in some cases you will encounter relations for which there's really no other way to get the result. That is why you will find that in physics applications where you might use sines or cosines (oscillations, waves) no one uses anything but complex exponentials. Get used to it.

The trigonometric functions of complex argument follow naturally from these.

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \text{so, for negative angle} \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

Add these and subtract these to get

$$\cos\theta = \frac{1}{2}\big(e^{i\theta} + e^{-i\theta}\big) \qquad \text{and} \qquad \sin\theta = \frac{1}{2i}\big(e^{i\theta} - e^{-i\theta}\big) \tag{3.9}$$

What is this if $\theta = iy$?

$$\cos iy = \frac{1}{2}\big(e^{-y} + e^{+y}\big) = \cosh y \qquad \text{and} \qquad \sin iy = \frac{1}{2i}\big(e^{-y} - e^{+y}\big) = i\sinh y \tag{3.10}$$

Apply Eq. (3.8) for the addition of angles to the case that $\theta = x + iy$.

$$\begin{aligned}&\cos(x+iy) = \cos x\cos iy - \sin x\sin iy = \cos x\cosh y - i\sin x\sinh y \qquad \text{and} \\ &\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y\end{aligned} \tag{3.11}$$

You can see from this that the sine and cosine of complex angles can be real and larger than one. The hyperbolic functions and the circular trigonometric functions are now the same functions. You're just looking in two different directions in the complex plane. It's as if you are changing from the equation of a circle, $x^2 + y^2 = R^2$, to that of a hyperbola, $x^2 - y^2 = R^2$. Compare this to the hyperbolic functions at the beginning of chapter one.

Equation (3.9) doesn't require that $\theta$ itself be real; call it $z$. Then what is $\sin^2 z + \cos^2 z$?

$$\cos z = \frac{1}{2}\big(e^{iz} + e^{-iz}\big) \qquad \text{and} \qquad \sin z = \frac{1}{2i}\big(e^{iz} - e^{-iz}\big)$$

$$\cos^2 z + \sin^2 z = \frac{1}{4}\big[e^{2iz} + e^{-2iz} + 2 - e^{2iz} - e^{-2iz} + 2\big] = 1$$

This polar form shows a geometric interpretation for the periodicity of the exponential. $e^{i(\theta+2\pi)} = e^{i\theta} = e^{i(\theta+2k\pi)}$. In the picture, you're going around a circle and coming back to the same point. If the angle $\theta$ is negative you're just going around in the opposite direction. An angle of $-\pi$ takes you to the same point as an angle of $+\pi$.

**Complex Conjugate**

The complex conjugate of a number $z = x + iy$ is the number $z^* = x - iy$. Another common notation is $\bar{z}$. The product $z^*z$ is $(x-iy)(x+iy) = x^2+y^2$ and that is $|z|^2$, the square of the magnitude of $z$. You can use this to rearrange complex fractions, combining the various terms with $i$ in them and putting them in one place. This is best shown by some examples.

$$\frac{3+5i}{2+3i} = \frac{(3+5i)(2-3i)}{(2+3i)(2-3i)} = \frac{21+i}{13}$$

What happens when you add the complex conjugate of a number to the number, $z + z^*$?
What happens when you subtract the complex conjugate of a number from the number?
If one number is the complex conjugate of another, how do their squares compare?
What about their cubes?
What about $z + z^2$ and $z^* + z^{*2}$?
What about comparing $e^z = e^{x+iy}$ and $e^{z^*}$?
What is the product of a number and its complex conjugate written in polar form?
Compare $\cos z$ and $\cos z^*$.
What is the quotient of a number and its complex conjugate?
What about the magnitude of the preceding quotient?

**Examples**

Simplify these expressions, making sure that you can do all of these manipulations yourself.

$$\frac{3-4i}{2-i} = \frac{(3-4i)(2+i)}{(2-i)(2+i)} = \frac{10-5i}{5} = 2-i.$$

$$(3i+1)^2\left[\frac{1}{2-i}+\frac{3i}{2+i}\right] = (-8+6i)\left[\frac{(2+i)+3i(2-i)}{(2-i)(2+i)}\right] = (-8+6i)\frac{5+7i}{5} = \frac{2-26i}{5}.$$

$$\frac{i^3+i^{10}+i}{i^2+i^{137}+1} = \frac{(-i)+(-1)+i}{(-1)+(i)+(1)} = \frac{-1}{i} = i.$$

Manipulate these using the polar form of the numbers, though in some cases you can do it either way.

$$\sqrt{i} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}.$$

$$\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{\sqrt{2}e^{-i\pi/4}}{\sqrt{2}e^{i\pi/4}}\right)^3 = \left(e^{-i\pi/2}\right)^3 = e^{-3i\pi/2} = i.$$

$$\left(\frac{2i}{1+i\sqrt{3}}\right)^{25} = \left(\frac{2e^{i\pi/2}}{2\left(\frac{1}{2}+i\frac{1}{2}\sqrt{3}\right)}\right)^{25} = \left(\frac{2e^{i\pi/2}}{2e^{i\pi/3}}\right)^{25} = \left(e^{i\pi/6}\right)^{25} = e^{i\pi(4+1/6)} = \tfrac{1}{2}\left(\sqrt{3}+i\right)$$

**Roots of Unity**

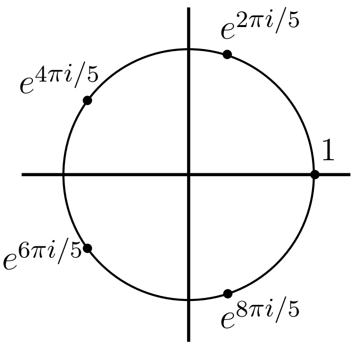
What is the cube root of one? One of course, but not so fast; there are three cube roots, and you can easily find all of them using complex exponentials.

$$1 = e^{2k\pi i}, \qquad \text{so} \qquad 1^{1/3} = \left(e^{2k\pi i}\right)^{1/3} = e^{2k\pi i/3} \tag{3.12}$$

and $k$ is any integer. $k = 0, 1, 2$ give

$$1^{1/3} = 1, \qquad e^{2\pi i/3} = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad e^{4\pi i/3} = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

and other positive or negative integers $k$ just keep repeating these three values.




$5^{th}$ roots of 1

The roots are equally spaced around the unit circle. If you want the $n^{\text{th}}$ root, you do the same sort of calculation: the $1/n$ power and the integers $k = 0, 1, 2, \ldots, (n-1)$. These are $n$ points, and the angles between adjacent ones are equal.

### 3.4 Geometry

Multiply a number by 2 and you change its length by that factor.
Multiply it by $i$ and you rotate it counterclockwise by $90^\circ$ about the origin.
Multiply is by $i^2 = -1$ and you rotate it by $180^\circ$ about the origin. (Either direction: $i^2 = (-i)^2$)

The Pythagorean Theorem states that if you construct three squares from the three sides of a right triangle, the sum of the two areas on the shorter sides equals the area of the square constructed on the hypotenuse. What happens if you construct four squares on the four sides of an arbitrary quadrilateral?

Represent the four sides of the quadrilateral by four complex numbers that add to zero. Start from the origin and follow the complex number $a$. Then follow $b$, then $c$, then $d$. The result brings you back to the origin. Place four squares on the four sides and locate the centers of those squares: $P_1$, $P_2$,... Draw lines between these points as shown.

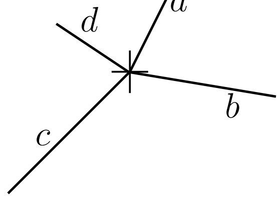
These lines are orthogonal and have the same length. Stated in the language of complex numbers, this is

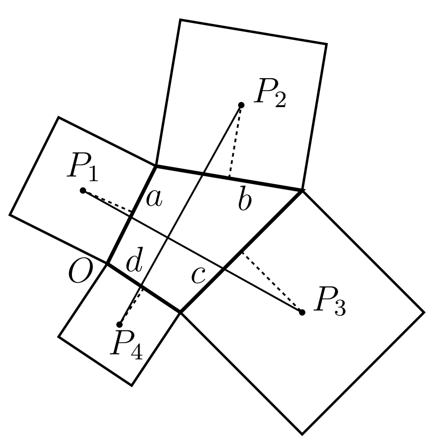
$$P_1 - P_3 = i\big(P_2 - P_4\big) \tag{3.13}$$

$$a + b + c + d = 0$$

$$\tfrac{1}{2}a + \tfrac{1}{2}ia = P_1$$

$$a + \tfrac{1}{2}b + \tfrac{1}{2}ib = P_2$$




Pick the origin at one corner, then construct the four center points $P_{1,2,3,4}$ as complex numbers, following the pattern shown above for the first two. *E.g.* , you get to $P_1$ from the origin by going halfway along $a$, turning left, then going the distance $|a|/2$. Now write out the two complex number $P_1 - P_3$ and $P_2 - P_4$ and finally manipulate them by using the defining equation for the quadrilateral, $a+b+c+d=0$. The result is the stated theorem. See problem 3.54.

### 3.5 Series of cosines

There are standard identities for the cosine and sine of the sum of angles and less familiar ones for the sum of two cosines or sines. You can derive that latter sort of equations using Euler's formula and a little manipulation. The sum of two cosines is the real part of $e^{ix} + e^{iy}$, and you can use simple identities to manipulate these into a useful form.

$$x = \tfrac{1}{2}(x+y) + \tfrac{1}{2}(x-y) \qquad \text{and} \qquad y = \tfrac{1}{2}(x+y) - \tfrac{1}{2}(x-y)$$

See problems 3.34 and 3.35 to complete these.

What if you have a sum of many cosines or sines? Use the same basic ideas of the preceding manipulations, and combine them with some of the techniques for manipulating series.

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos N\theta = 1 + e^{i\theta} + e^{2i\theta} + \cdots e^{Ni\theta} \qquad \text{(Real part)}$$

The last series is geometric, so it is nothing more than Eq. (2.3).

$$\begin{aligned} 1 + e^{i\theta} + \left(e^{i\theta}\right)^2 + \left(e^{i\theta}\right)^3 + \cdots \left(e^{i\theta}\right)^N &= \frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}} \\ = \frac{e^{i(N+1)\theta/2}\left(e^{-i(N+1)\theta/2} - e^{i(N+1)\theta/2}\right)}{e^{i\theta/2}\left(e^{-i\theta/2} - e^{i\theta/2}\right)} &= e^{iN\theta/2}\frac{\sin\left[(N+1)\theta/2\right]}{\sin\theta/2} \end{aligned} \tag{3.14}$$

From this you now extract the real part and the imaginary part, thereby obtaining the series you want (plus another one, the series of sines). These series appear when you analyze the behavior of a diffraction grating. Naturally you have to check the plausibility of these results; do the answers work for small $\theta$?

### 3.6 Logarithms

The logarithm is the inverse function for the exponential. If $e^w = z$ then $w = \ln z$. To determine what this is, let

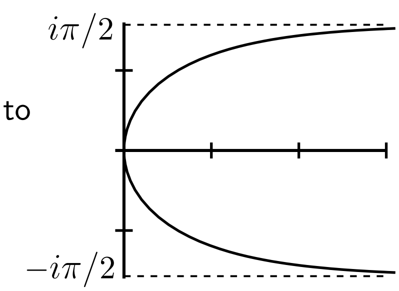
$$w = u + iv \qquad \text{and} \qquad z = re^{i\theta}, \qquad \text{then} \qquad e^{u+iv} = e^u e^{iv} = re^{i\theta}$$

This implies that $e^u = r$ and so $u = \ln r$, but it doesn't imply $v = \theta$. Remember the periodic nature of the exponential function? $e^{i\theta} = e^{i(\theta+2n\pi)}$, so you can conclude instead that $v = \theta + 2n\pi$.

$$\ln z = \ln\left(re^{i\theta}\right) = \ln r + i(\theta + 2n\pi) \tag{3.15}$$

has an infinite number of possible values. Is this bad? You're already familiar with the square root function, and that has *two* possible values, $\pm$. This just carries the idea farther. For example $\ln(-1) = i\pi$ or $3i\pi$ or $-7i\pi$ etc. As with the square root, the specific problem that you're dealing with will tell you which choice to make.

A sample graph of the logarithm in the complex plane is $\ln(1+it)$ as $t$ varies from $-\infty$ to $+\infty$.

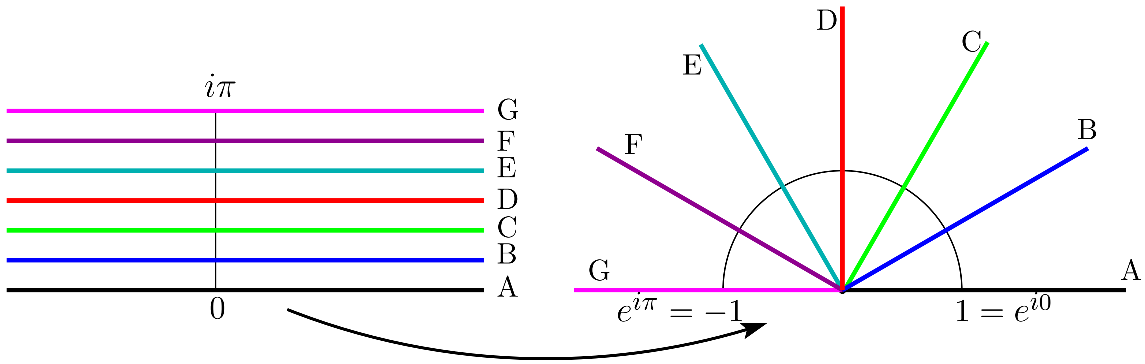


### 3.7 Mapping

When you apply a complex function to a region in the plane, it takes that region into another region. When you look at this as a geometric problem you start to get some very pretty and occasionally useful results. Start with a simple example,

$$w = f(z) = e^z = e^{x+iy} = e^x e^{iy} \tag{3.16}$$

If $y = 0$ and $x$ goes from $-\infty$ to $+\infty$, this function goes from 0 to $\infty$.
If $y$ is $\pi/4$ and $x$ goes over this same range of values, $f$ goes from 0 to infinity along the ray at angle $\pi/4$ above the axis.
At any fixed $y$, the horizontal line parallel to the $x$-axis is mapped to the ray that starts at the origin and goes out to infinity.
The strip from $-\infty < x < +\infty$ and $0 < y < \pi$ is mapped into the upper half plane.

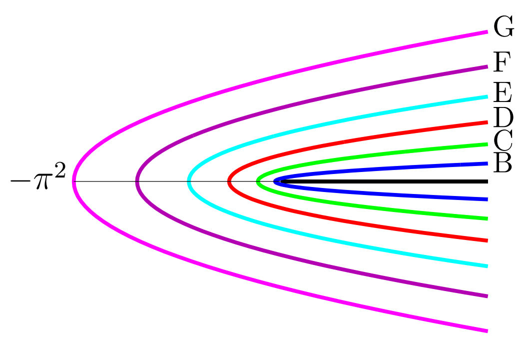



The line B from $-\infty + i\pi/6$ to $+\infty + i\pi/6$ is mapped onto the ray B from the origin along the angle $\pi/6$.

For comparison, what is the image of the same strip under a different function? Try

$$w = f(z) = z^2 = x^2 - y^2 + 2ixy$$

The image of the line of fixed $y$ is a parabola. The real part of $w$ has an $x^2$ in it while the imaginary part is linear in $x$. That is the representation of a parabola. The image of the strip is the region among the lines below.




Pretty yes, but useful? In certain problems in electrostatics and in fluid flow, it is possible to use complex algebra to map one region into another, with the accompanying electric fields and potentials or respectively fluid flows mapped from a complicated problem into a simple one. Then you can map the simple solution back to the original problem and you have your desired solution to the original problem. Easier said than done. It's the sort of method that you can learn about when you find that you need it.

## Exercises

**1** Express in the form $a + ib$: $(3 - i)^2$, $(2 - 3i)(3 + 4i)$. Draw the geometric representation for each calculation.

**2** Express in polar form, $re^{i\theta}$: $-2$, $3i$, $3 + 3i$. Draw the geometric representation for each.

**3** Show that $(1 + 2i)(3 + 4i)(5 + 6i)$ satisfies the associative law of multiplication. *I.e.* multiply first pair first or multiply the second pair first, no matter.

**4** Solve the equation $z^2 - 2z + c = 0$ and plot the roots as points in the complex plane. Do this as the real number $c$ moves from $c = 0$ to $c = 2$

**5** Now show that $(a + bi)\big[(c + di)(e + fi)\big] = \big[(a + bi)(c + di)\big](e + fi)$. After all, just because real numbers satisfy the associative law of multiplication it isn't immediately obvious that complex numbers do too.

**6** Given $z_1 = 2e^{i60^\circ}$ and $z_2 = 4e^{i120^\circ}$, evaluate $z_1^2$, $z_1 z_2$, $z_2/z_1$. Draw pictures too.

**7** Evaluate $\sqrt{i}$ using the rectangular form, Eq. (3.2), and compare it to the result you get by using the polar form.

**8** Given $f(z) = z^2 + z + 1$, evaluate $f(3 + 2i)$, $f(3 - 2i)$.

**9** For the same $f$ as the preceding exercise, what are $f'(3 + 2i)$ and $f'(3 - 2i)$?

**10** Do the arithmetic and draw the pictures of these computations:

$$(3 + 2i) + (-1 + i), \qquad (3 + 2i) - (-1 + i), \qquad (-4 + 3i) - (4 + i), \qquad -5 + (3 - 5i)$$

**11** Show that the real part of $z$ is $(z + z^*)/2$. Find a similar expression for the imaginary part of $z$.

**12** What is $i^n$ for integer $n$? Draw the points in the complex plane for a variety of positive and negative $n$.

**13** What is the magnitude of $(4 + 3i)/(3 - 4i)$? What is its polar angle?

**14** Evaluate $(1 + i)^{19}$.

**15** What is $\sqrt{1 - i}$? Do this by the method of Eq. (3.2).

**16** What is $\sqrt{1 - i}$? Do this by the method of Eq. (3.6).

**17** Sketch a plot of the curve $z = \alpha e^{i\alpha}$ as the real parameter $\alpha$ varies from zero to infinity. Does the behavior of your sketch conform to the small $\alpha$ behavior of the function? (And when no one's looking you can plug in a few numbers for $\alpha$ to see what this behavior is.)

**18** Verify the graph following Eq. (3.15).

**Problems**

**3.1** Pick a pair of complex numbers and plot them in the plane. Compute their product and plot that point. Do this for several pairs, trying to get a feel for how complex multiplication works. When you do this, be sure that you're not simply repeating yourself. Place the numbers in qualitatively different places.

**3.2** In the calculation of the square root of a complex number,Eq. (3.2), I found four roots instead of two. Which ones don't belong? Do the other two expressions have any meaning?

**3.3** Finish the algebra in computing the reciprocal of a complex number, Eq. (3.3).

**3.4** Pick a complex number and plot it in the plane. Compute its reciprocal and plot it. Compute its square and square root and plot them. Do this for several more (qualitatively different) examples.

**3.5** Plot $e^{ct}$ in the plane where $c$ is a complex constant of your choosing and the parameter $t$ varies over $0 \le t < \infty$. Pick another couple of values for $c$ to see how the resulting curves change. Don't pick values that simply give results that are qualitatively the same; pick values sufficiently varied so that you can get different behavior. If in doubt about how to plot these complex numbers as functions of $t$, pick a few numerical values: *e.g.* $t = 0.01, 0.1,\ 0.2,\ 0.3$, *etc.* Ans: Spirals or straight lines, depending on where you start

**3.6** Plot $\sin ct$ in the plane where $c$ is a complex constant of your choosing and the parameter $t$ varies over $0 \le t < \infty$. Pick another couple of qualitatively different values for $c$ to see how the resulting curves change.

**3.7** Solve the equation $z^2 + iz + 1 = 0$

**3.8** Just as Eq. (3.11) presents the circular functions of complex arguments, what are the hyperbolic functions of complex arguments?

**3.9** From $\left(e^{ix}\right)^3$, deduce trigonometric identities for the cosine and sine of triple angles in terms of single angles. Ans: $\cos 3x = \cos x - 4\sin^2 x \cos x = 4\cos^3 x - 3\cos x$

**3.10** For arbitrary integer $n > 1$, compute the sum of all the $n^{\text{th}}$ roots of one. (When in doubt, try $n = 2,\ 3,\ 4$ first.)

**3.11** Either solve for $z$ in the equation $e^z = 0$ or prove that it can't be done.

**3.12** Evaluate $z/z^*$ in polar form.

**3.13** From the geometric picture of the magnitude of a complex number, the set of points $z$ defined by $|z - z_0| = R$ is a circle. Write it out in rectangular components to see what this is in conventional Cartesian coordinates.

**3.14** An ellipse is the set of points $z$ such that the sum of the distances to two fixed points is a constant: $|z - z_1| + |z - z_2| = 2a$. Pick the two points to be $z_1 = -f$ and $z_2 = +f$ on the real axis ($f < a$). Write $z$ as $x + iy$ and manipulate this equation for the ellipse into a simple standard form. I suggest that you leave everything in terms of complex numbers ($z$, $z^*$, $z_1$, $z_1^*$, *etc.* ) until some distance into the problem. Use $x + iy$ only after it becomes truly useful to do so.

**3.15** Repeat the previous problem, but for the set of points such that the *difference* of the distances from two fixed points is a constant.

**3.16** There is a vertical line $x = -f$ and a point on the $x$-axis $z_0 = +f$. Find the set of points $z$ so that the distance to $z_0$ is the same as the perpendicular distance to the line $x = -f$.

**3.17** Sketch the set of points $|z - 1| < 1$.

**3.18** Simplify the numbers

$$\frac{1+i}{1-i}, \qquad \frac{-1+i\sqrt{3}}{+1+i\sqrt{3}}, \qquad \frac{i^5+i^3}{\sqrt{3\sqrt{i}-7\sqrt[3]{17-4i}}}, \qquad \left(\frac{\sqrt{3}+i}{1+i}\right)^2$$

**3.19** Express in polar form; include a sketch in each case.

$$2-2i, \qquad \sqrt{3}+i, \qquad -\sqrt{5}i, \qquad -17-23i$$

**3.20** Take two complex numbers; express them in polar form, and subtract them.

$$z_1 = r_1 e^{i\theta_1}, \qquad z_2 = r_2 e^{i\theta_2}, \qquad \text{and} \qquad z_3 = z_2 - z_1$$

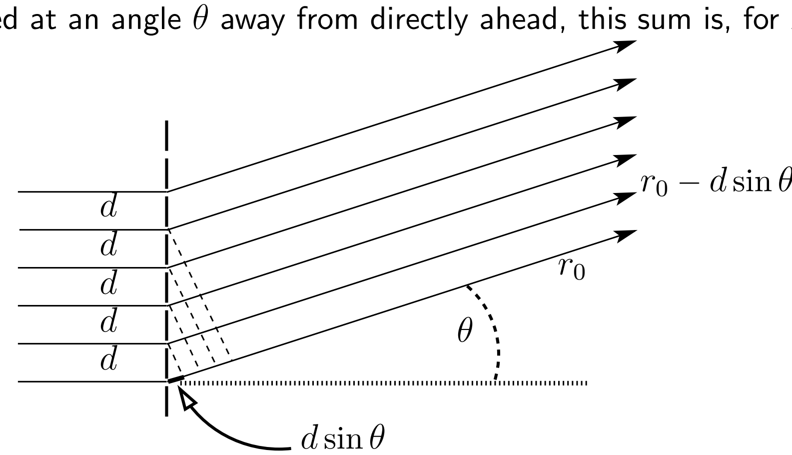
Compute $z_3^* z_3$, the magnitude squared of $z_3$, and so derive the law of cosines. You *did* draw a picture didn't you?

**3.21** What is $i^i$? Ans: If you'd like to check your result, type $i \wedge i$ into Google. Or use a calculator such as the one mentioned on page 6.

**3.22** For what argument does $\sin\theta = 2$? Next: $\cos\theta = 2$?
Ans: $\sin^{-1} 2 = 1.5708 \pm i1.3170$

**3.23** What are the other trigonometric functions, $\tan(ix)$, $\sec(ix)$, etc. What are $\tan$ and $\sec$ for the general argument $x + iy$.
Ans: $\tan(x+iy) = (\tan x + i\tanh y)/(1 - i\tan x\tanh y)$

**3.24** The diffraction pattern from a grating involves the sum of waves from a large number of parallel slits. For light observed at an angle $\theta$ away from directly ahead, this sum is, for $N+1$ slits,



$$\cos\big(kr_0-\omega t\big)+\cos\big(k(r_0-d\sin\theta)-\omega t\big)+\cos\big(k(r_0-2d\sin\theta)-\omega t\big)+\\ \ldots+\cos\big(k(r_0-Nd\sin\theta)-\omega t\big)$$

Express this as the real part of complex exponentials and sum the finite series. Show that the resulting wave is

$$\frac{\sin\big(\frac{1}{2}(N+1)kd\sin\theta\big)}{\sin\big(\frac{1}{2}kd\sin\theta\big)}\cos\big(k(r_0-\tfrac{1}{2}Nd\sin\theta)-\omega t\big)$$

Interpret this result as a wave that appears to be coming from some particular point (where?) and with an intensity pattern that varies strongly with $\theta$.

**3.25** (**a**) If the coefficients in a quadratic equation are real, show that if $z$ is a complex root of the equation then so is $z^*$. If you do this by reference to the quadratic formula, you'd better find another way too, because the second part of this problem is
(**b**) Generalize this to the roots of an arbitrary polynomial with real coefficients.

**3.26** You can represent the motion of a particle in two dimensions by using a time-dependent complex number with $z=x+iy=re^{i\theta}$ showing its rectangular or polar coordinates. Assume that $r$ and $\theta$ are functions of time and differentiate $re^{i\theta}$ to get the velocity. Differentiate it again to get the acceleration. You can interpret $e^{i\theta}$ as the unit vector along the radius and $ie^{i\theta}$ as the unit vector perpendicular to the radius and pointing in the direction of increasing theta. Show that

$$\frac{d^2z}{dt^2}=e^{i\theta}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]+ie^{i\theta}\left[r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}\right] \tag{3.17}$$

and translate this into the usual language of components of vectors, getting the radial ($\hat{r}$) component of acceleration and the angular component of acceleration as in section 8.9.

**3.27** Use the results of the preceding problem, and examine the case of a particle moving directly away from the origin. (**a**) What is its acceleration? (**b**) If instead, it is moving at $r=$constant, what is its acceleration? (**c**) If instead, $x=x_0$ and $y=v_0t$, what are $r(t)$ and $\theta(t)$? Now compute $d^2z/dt^2$ from Eq. (3.17).

**3.28** Was it really legitimate simply to substitute $x+iy$ for $\theta_1+\theta_2$ in Eq. (3.11) to get $\cos(x+iy)$? Verify the result by substituting the expressions for $\cos x$ and for $\cosh y$ as exponentials to see if you can reconstruct the left-hand side.

**3.29** The roots of the quadratic equation $z^2+bz+c=0$ are functions of the parameters $b$ and $c$. For real $b$ and $c$ and for both cases $c>0$ and $c<0$ (say $\pm1$ to be specific) plot the trajectories of the roots in the complex plane as $b$ varies from $-\infty$ to $+\infty$. You should find various combinations of straight lines and arcs of circles.

**3.30** In integral tables you can find the integrals for such functions as

$$\int dx\,e^{ax}\cos bx, \qquad \text{or} \qquad \int dx\,e^{ax}\sin bx$$

Show how easy it is to do these by doing both integrals at once. Do the first plus $i$ times the second and then separate the real and imaginary parts.

**3.31** Find the sum of the series
$$\sum_1^\infty \frac{i^n}{n}$$
Ans: $i\pi/4 - \frac{1}{2}\ln 2$

**3.32** Evaluate $|\cos z|^2$. Evaluate $|\sin z|^2$.

**3.33** Evaluate $\sqrt{1+i}$. Evaluate $\ln(1+i)$. Evaluate $\tan(1+i)$.

**3.34** (**a**) Beats occur in sound when two sources emit two frequencies that are almost the same. The perceived wave is the sum of the two waves, so that at your ear, the wave is a sum of two cosines of $\omega_1 t$ and of $\omega_2 t$. Use complex algebra to evaluate this. The sum is the real part of
$$e^{i\omega_1 t} + e^{i\omega_2 t}$$
Notice the two identities
$$\omega_1 = \frac{\omega_1 + \omega_2}{2} + \frac{\omega_1 - \omega_2}{2}$$
and the difference of these for $\omega_2$. Use the complex exponentials to derive the results; don't just look up some trig identity. Factor the resulting expression and sketch a graph of the resulting real part, interpreting the result in terms of beats if the two frequencies are close to each other. (**b**) In the process of doing this problem using complex exponentials, what is the trigonometric identity for the sum of two cosines? While you're about it, what is the difference of two cosines?
Ans: $\cos\omega_1 t + \cos\omega_2 t = 2\cos\frac{1}{2}(\omega_1+\omega_2)t\cos\frac{1}{2}(\omega_1-\omega_2)t$

**3.35** Derive using complex exponentials: $\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$.

**3.36** The equation (3.4) assumed that the usual rule for multiplying exponentials still holds when you are using complex numbers. Does it? You can prove it by looking at the infinite series representation for the exponential and showing that
$$e^a e^b = \left[1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots\right]\left[1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \cdots\right] = \left[1 + (a+b) + \frac{(a+b)^2}{2!} + \cdots\right]$$
You may find Eq. (2.19) useful.

**3.37** Look at the vertical lines in the $z$-plane as mapped by Eq. (3.16). I drew the images of lines $y =$constant, now you draw the images of the straight line segments $x =$constant from $0 < y < \pi$. The two sets of lines in the original plane intersect at right angles. What is the angle of intersection of the corresponding curves in the image?

**3.38** Instead of drawing the image of the lines $x =$constant as in the previous problem, draw the image of the line $y = x\tan\alpha$, the line that makes an angle $\alpha$ with the horizontal lines. The image of the horizontal lines were radial lines. At a point where this curve intersects one of the radial lines, what angle does the curve make with the radial line? Show that the answer is $\alpha$, the same angle of intersection as in the original picture.

**3.39** Write each of these functions of $z$ as two real functions $u$ and $v$ such that $f(z) = u(x,y) + iv(x,y)$.
$$z^3, \qquad \frac{1+z}{1-z}, \qquad \frac{1}{z^2}, \qquad \frac{z}{z^*}$$

**3.40** Evaluate $z^i$ where $z$ is an arbitrary complex number, $z = x + iy = re^{i\theta}$.

**3.41** What is the image of the domain $-\infty < x < +\infty$ and $0 < y < \pi$ under the function $w = \sqrt{z}$? Ans: One boundary is a hyperbola.

**3.42** What is the image of the disk $|z - a| < b$ under the function $w = cz + d$? Allow $c$ and $d$ to be complex. Take $a$ real.

**3.43** What is the image of the disk $|z - a| < b$ under the function $w = 1/z$? Assume $b < a$. Ans: Another disk, centered at $a/(a^2 - b^2)$.

**3.44** (**a**) Multiply $(2 + i)(3 + i)$ and deduce the identity

$$\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$$

(**b**) Multiply $(5 + i)^4(-239 + i)$ and deduce

$$4\tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4$$

For (**b**) a sketch will help sort out some signs.
(**c**) Using the power series representation of the $\tan^{-1}$, Eq. (2.27), how many terms would it take to compute 100 digits of $\pi$ as $4\tan^{-1}1$? How many terms would it take using each of these two representations, (**a**) and (**b**), for $\pi$? Ans: Almost a googol versus respectively about 540 and a few more than 180 terms.

**3.45** Use Eq. (3.9) and look back at the development of Eq. (1.4) to find the $\sin^{-1}$ and $\cos^{-1}$ in terms of logarithms.

**3.46** Evaluate the integral $\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}\cos\beta x$ for fixed real $\alpha$ and $\beta$. Sketch a graph of the result versus $\beta$. Sketch a graph of the result versus $\alpha$, and why does the graph behave as it does? Notice the rate at which the result approaches zero as either $\alpha \to 0$ or $\alpha \to \infty$. The behavior is very different in the two cases. Ans: $e^{-\beta^2/4\alpha}\sqrt{\pi/\alpha}$

**3.47** Does the equation $\sin z = 0$ have any roots other than the real ones? How about the cosine? The tangent?

**3.48** Compute (**a**) $\sin^{-1} i$. (**b**) $\cos^{-1} i$. (**c**) $\tan^{-1} i$. (**d**) $\sinh^{-1} i$. Ans: $\sin^{-1} i = 0 + 0.881\, i$, $\cos^{-1} i = \pi/2 - 0.881\, i$.

**3.49** By writing

$$\frac{1}{1 + x^2} = \frac{i}{2}\left[\frac{1}{x + i} - \frac{1}{x - i}\right]$$

and integrating, check the equation

$$\int_0^1 \frac{dx}{1 + x^2} = \frac{\pi}{4}$$

**3.50** Solve the equations (**a**) $\cosh u = 0$ (**b**) $\tanh u = 2$ (**c**) $\operatorname{sech} u = 2i$
Ans: $\operatorname{sech}^{-1} 2i = 0.4812 - i1.5707$

**3.51** Solve the equations (**a**) $z - 2z^* = 1$ (**b**) $z^3 - 3z^2 + 4z = 2i$ after verifying that $1 + i$ is a root. Compare the result of problem 3.25.

**3.52** Confirm the plot of $\ln(1+iy)$ following Eq. (3.15). Also do the corresponding plots for $\ln(10+iy)$ and $\ln(100+iy)$. And what do these graphs look like if you take the other branches of the logarithm, with the $i(\theta+2n\pi)$?

**3.53** Check that the results of Eq. (3.14) for cosines and for sines give the correct results for small $\theta$? What about $\theta \to 2\pi$?

**3.54** Finish the calculation leading to Eq. (3.13), thereby proving that the two indicated lines have the same length and are perpendicular.

**3.55** In the same spirit as Eq. (3.13) concerning squares drawn on the sides of an arbitrary quadrilateral, start with an arbitrary triangle and draw equilateral triangles on each side. Find the centroids of each of the equilateral triangles and connect them. The result is an equilateral triangle. Recall: the centroid is one third the distance from the base to the vertex. [This one requires more algebra than the one in the text.] (Napoleon's Theorem)