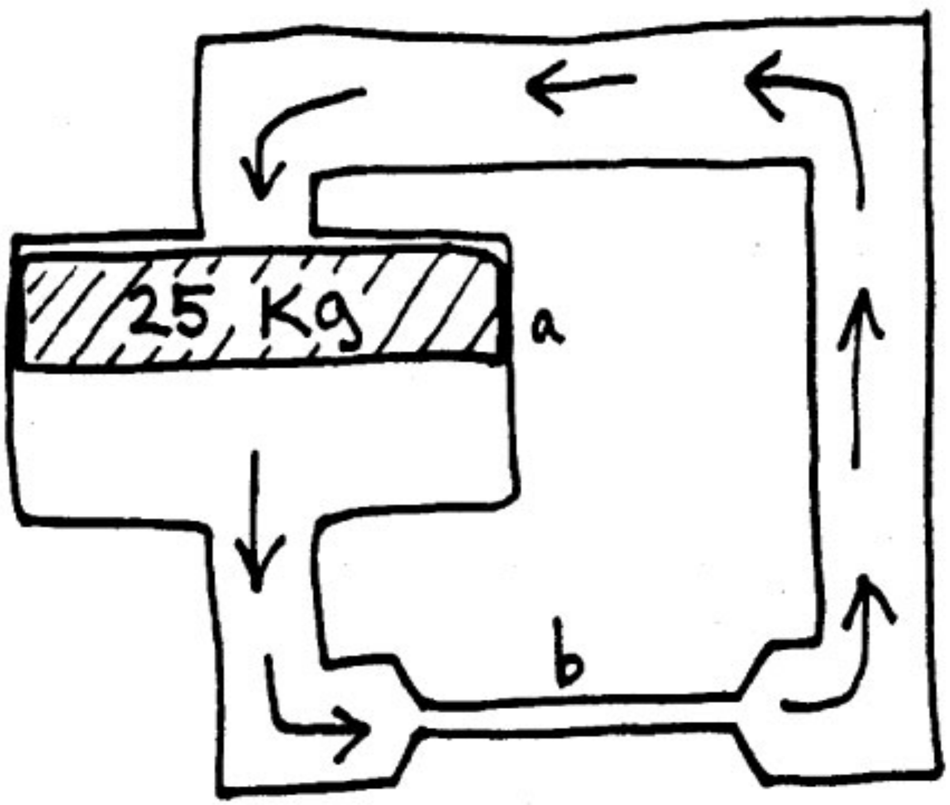
Alternating Current (AC) Circuits

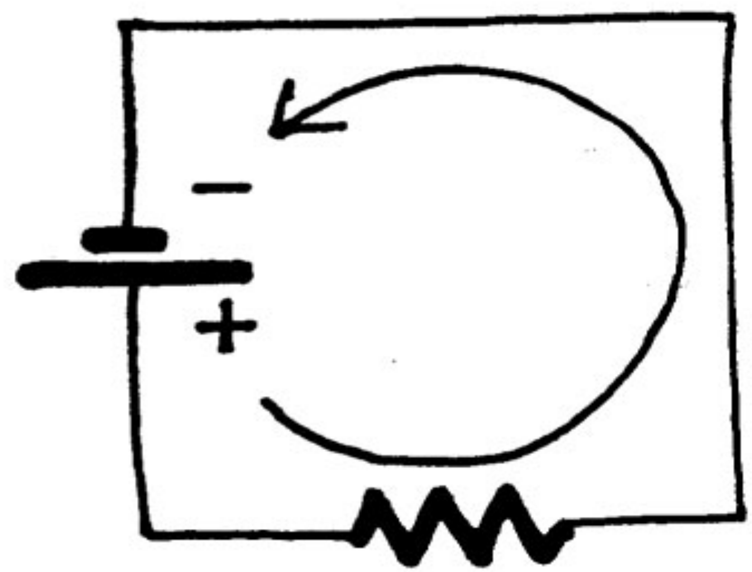
- We have been talking about DC circuits
  - Constant currents and voltages
  - Resistors
  - Linear equations

- Now we introduce AC circuits
  - Time-varying currents and voltages
  - Resistors, capacitors, inductors (coils)
  - Linear *differential* equations

Recall water analogy for Ohm's law…

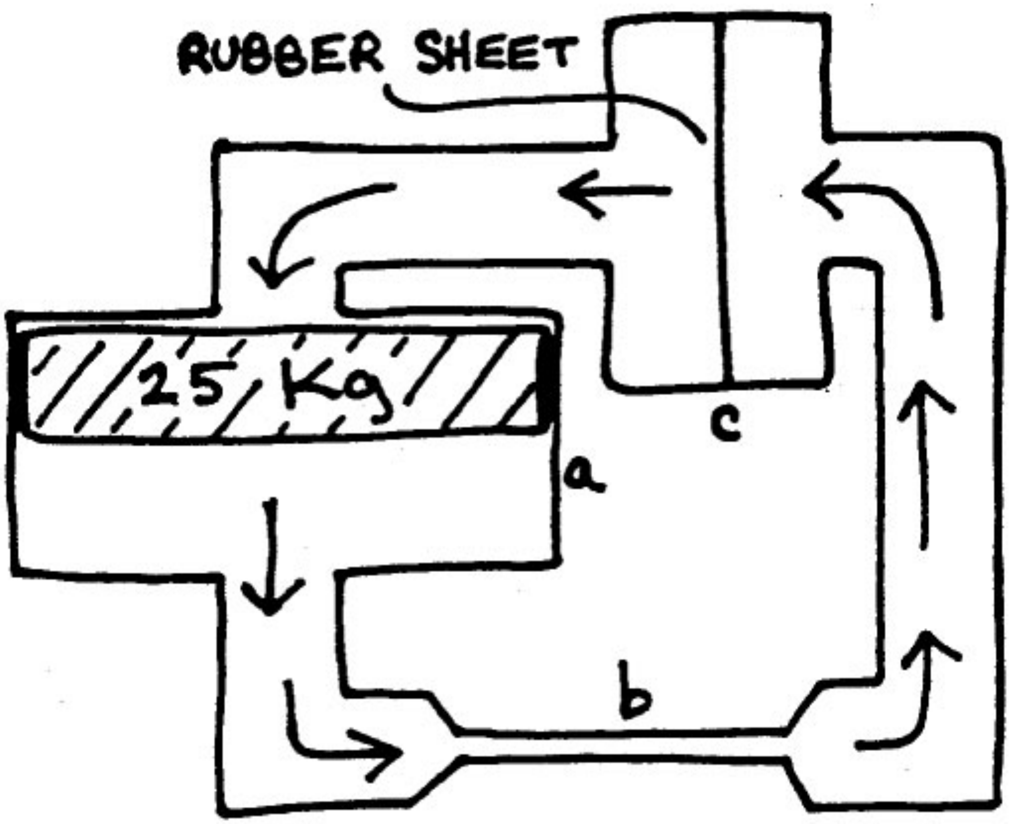
(a) Battery
(b) Resistor

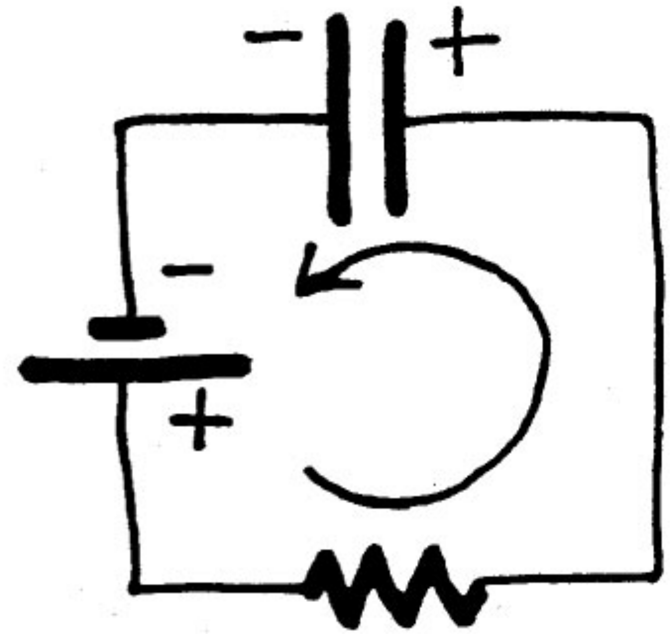




Now we add a steel tank with rubber sheet

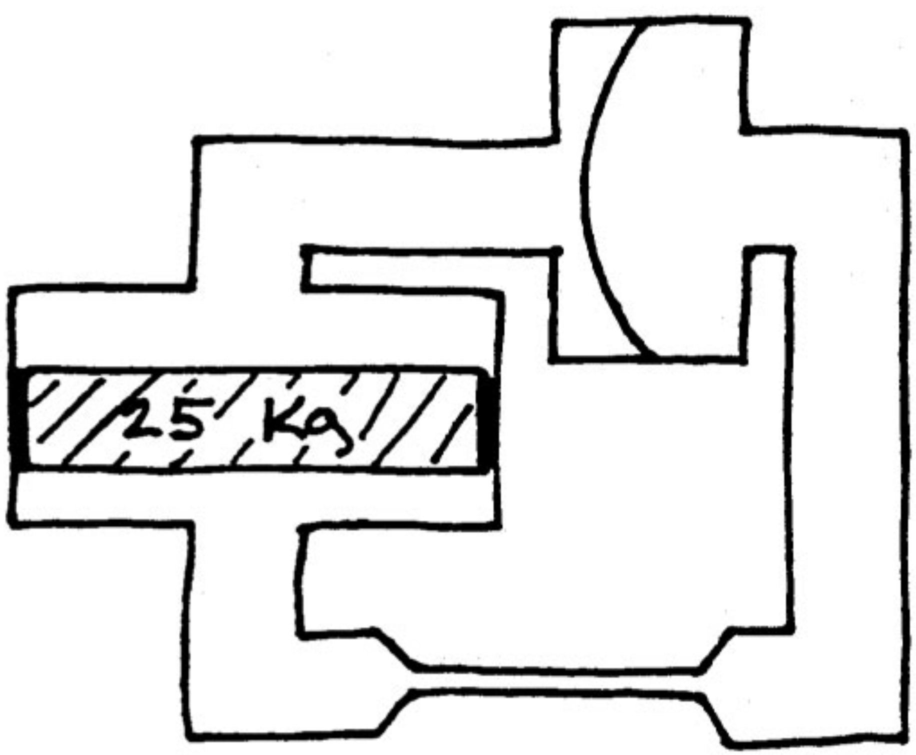
(a) Battery
(b) Resistor
(c) *Capacitor*

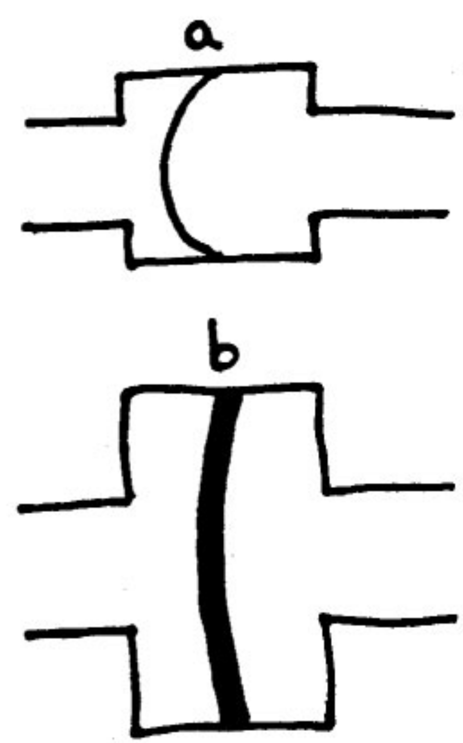




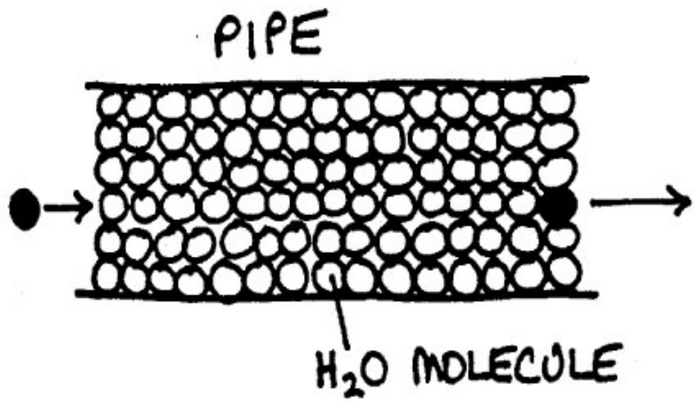
- Water enters one side of the tank and leaves the other, distending but not crossing the sheet.
- At first, water seems to flow *through* tank, but then pressure builds up pushing against the flow.
- How to decrease *capacitance* of tank?

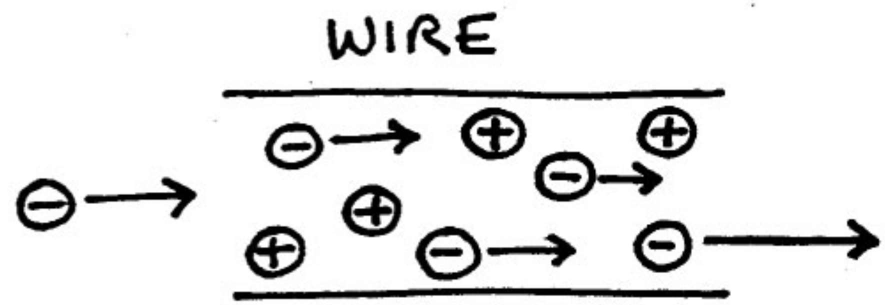
  Make rubber sheet (a) smaller or (b) thicker.



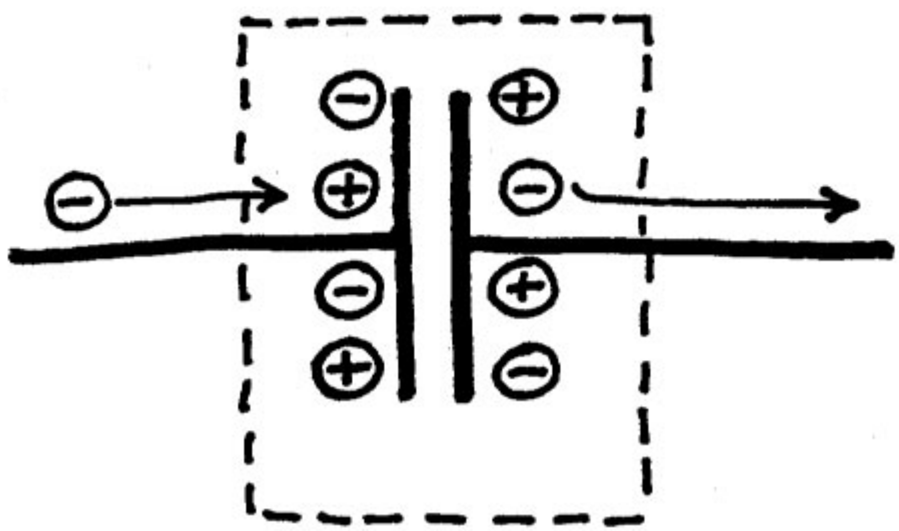


Charge, like water is practically incompressible,





but within a small volume (closely spaced plates) charge can enter one side and leave the other, without flowing across the space between.

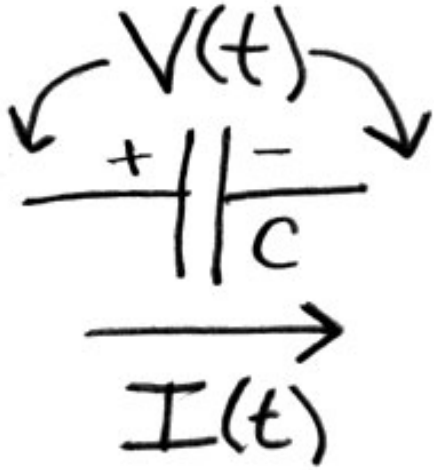


The apparent flow of current through space between the plates (the "displacement current") led Maxwell to discard the "ether" and derive equations governing EM waves.

Basic Laws of Capacitance

- *Capacitance C* relates charge *Q* to voltage *V*

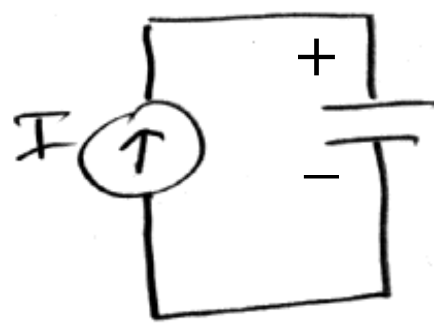
$$C = \frac{Q}{V}$$

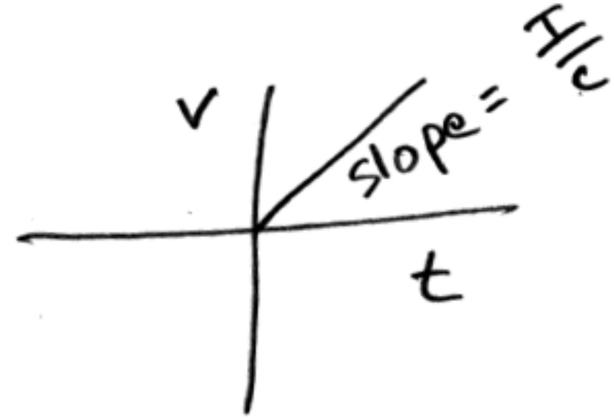


- Since $Q = \int I \, dt$,

$$V = \frac{1}{C} \int I \, dt$$

$$I = C \frac{dV}{dt}$$



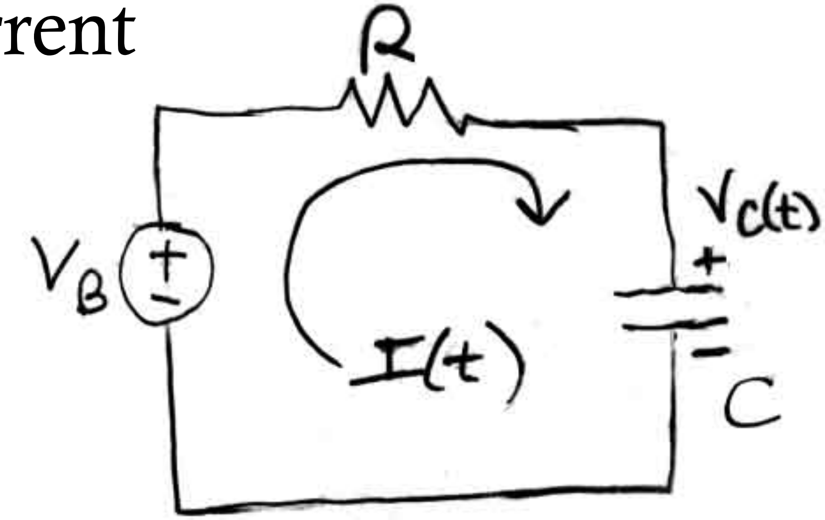


- Capacitance has units of *Farads*, F = 1 A sec / V

Charging a Capacitor with Battery V_B

- Voltage across resistor to find current

$$I(t) = \frac{V_\mathrm{B} - V_C(t)}{R}$$



- Basic law of capacitor

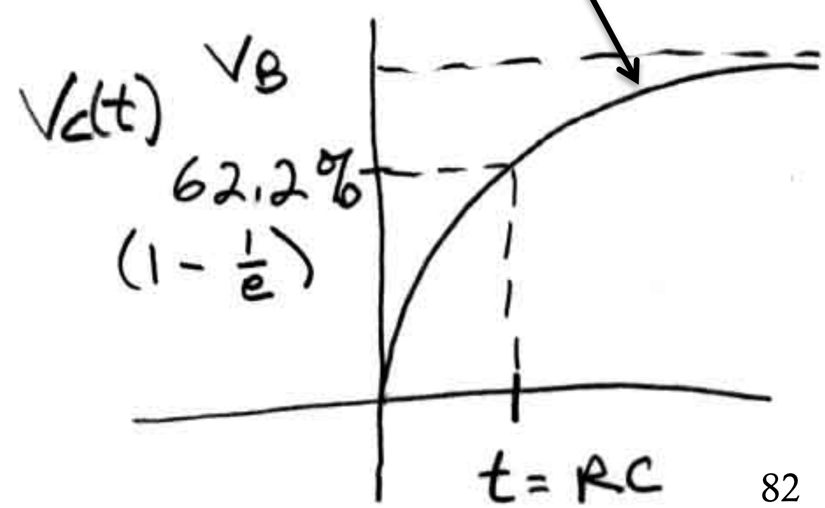
$$I(t) = C\frac{dV_\mathrm{C}(t)}{dt}$$

- Differential Equation yields exponential

$$V_\mathrm{C}(t) + RC\frac{dV_\mathrm{C}(t)}{dt} = V_\mathrm{B}$$

$$V_\mathrm{C}(t) = V_\mathrm{B}\left(1 - e^{-\frac{t}{RC}}\right)$$

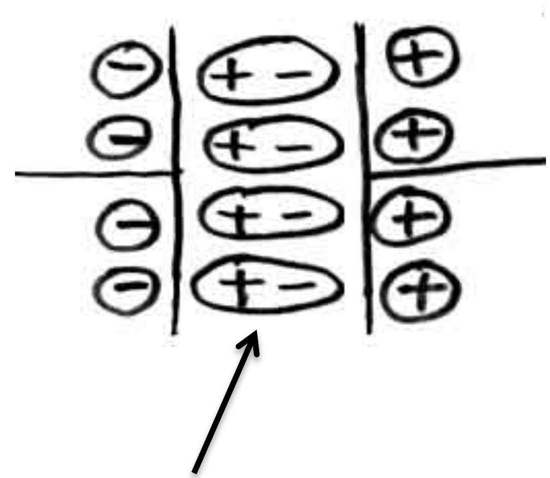
diminishing returns as cap becomes charged



What determines *capacitance* C?

- Area A of the plates
- Distance d between the plates
- *Permittivity* ε of the *dielectric* between plates.

$$C = \varepsilon \frac{A}{d}$$

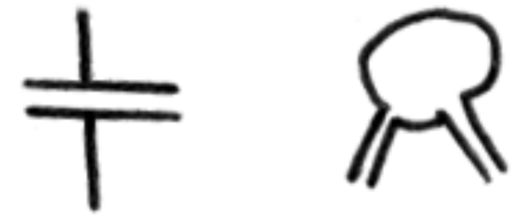


Alignment of dipoles within *dielectric* between plates increases capacitor's ability to store charge (capacitance).

Permittivity of a vacuum $\varepsilon_0 \approx 8.8541 \times 10^{-12} \mathrm{F} \cdot \mathrm{m}^{-1}$.

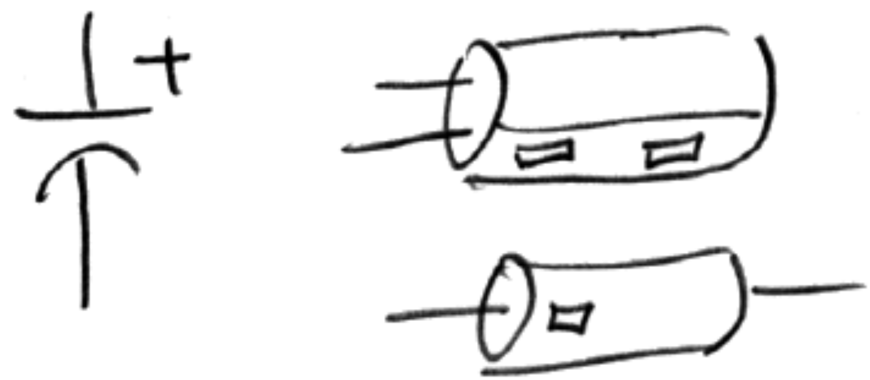
Types of Capacitors

- Disk (Ceramic) Capacitor
  - Non-polarized
  - Low leakage
  - High breakdown voltage
  - ~ 5pF – 0.1μF



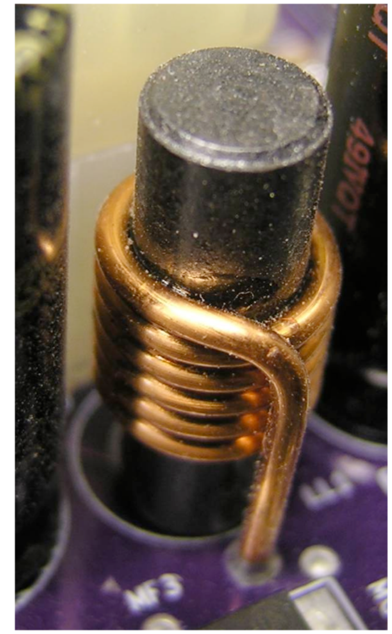
- 3 digits “ABC” = (AB plus C zeros)
  - “682” = 6800 pF
  - “104” = 100,000 pF = 0.1μF

- Electrolytic Capacitor
  - High leakage
  - Polarized
  - Low breakdown voltage
  - ~ 0.1μF – 10,000μF

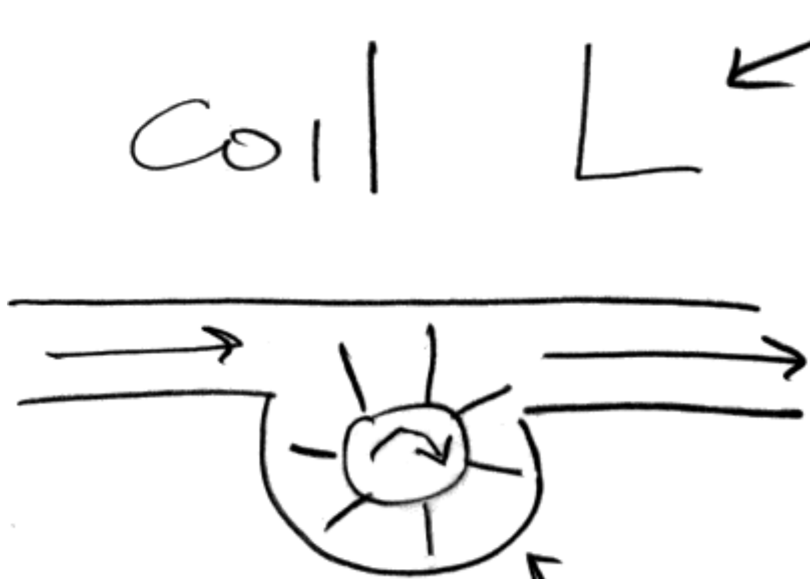


- Supercapacitor (Electrochemical Double Layer)
  - New. Effective spacing between plates in nanometers.
  - Many Farads! May power cars someday.

Inductor (coil)



- Water Analogy



inductance is like inertia/momentum of water in pipe with flywheel.

heavier flywheel
(coil wrapped around iron core)
adds to inertia/momentum.

Joseph Henry

1797 – 1878

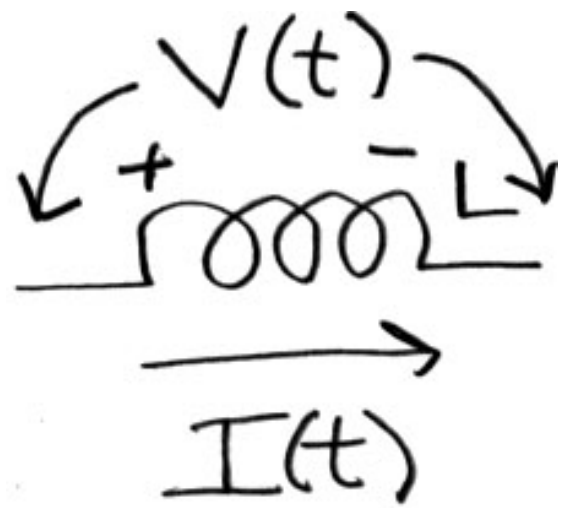
- Invented insulation
- Permitted construction of much more powerful electromagnets.
- Derived mathematics for "self-inductance"
- Built early relays, used to give telegraph range
- Put Princeton Physics on the map

Basic Laws of Inductance

- *Inductance L* relates changes in the current to voltages induced by changes in the magnetic field produced by the current.

$$I = \frac{1}{L}\int V\,dt$$

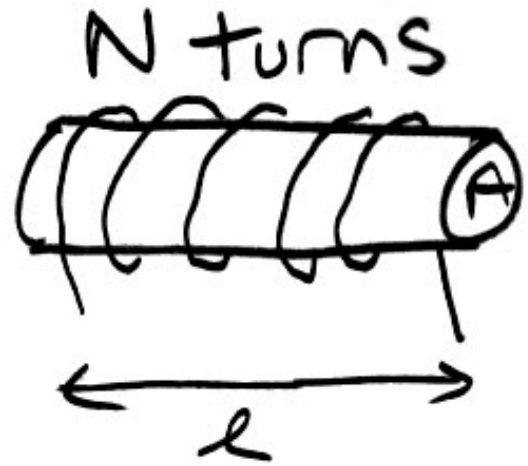
$$V = L\frac{dI}{dt}$$



- Inductance has units of *Henries,* H = 1 V sec / A.

What determines *inductance L* ?

- Assume a solenoid (coil)
- Area A of the coil
- Number of turns N
- Length ℓ of the coil
- *Permeability* μ of the core



$$L = \mu \frac{N^2 A}{\ell}$$

Permeability of a vacuum $\mu_0 \approx 1.2566 \times 10^{-6} \mathrm{H} \cdot \mathrm{m}^{-1}$.

Energy Stored in Capacitor

$$I = C\frac{dV}{dt}$$

$$P = VI = VC\frac{dV}{dt}$$

$$E = \int P\,dt$$

$$E = C\int V\,dV$$

$$E = \frac{1}{2}CV^2$$

Energy Stored in Caps and Coils

- Capacitors store "potential" energy in electric field

$$E = \frac{1}{2}CV^2$$ independent of history

- Inductors store "kinetic" energy in magnetic field

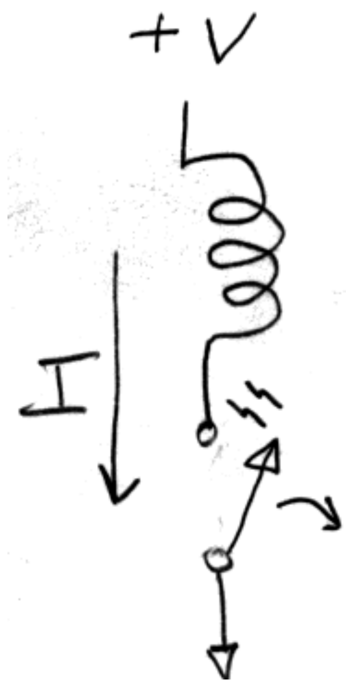
$$E = \frac{1}{2}LI^2$$ independent of history

- Resistors don't store energy at all!

the energy is dissipated as heat = $V \times I$

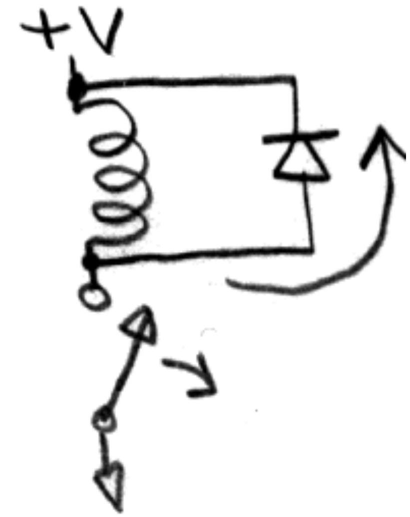
Generating Sparks

- What if you suddenly try to stop a current?



$$V = L\frac{dI}{dt}$$

goes to - ∞ when switch is opened.

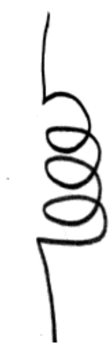


use diode to shunt current, protect switch.

- Nothing changes instantly in Nature.
- Spark coil used in early radio (Titanic).
- Tesla patented the spark plug.

Symmetry of Electromagnetism
(from an electronics component point of view)

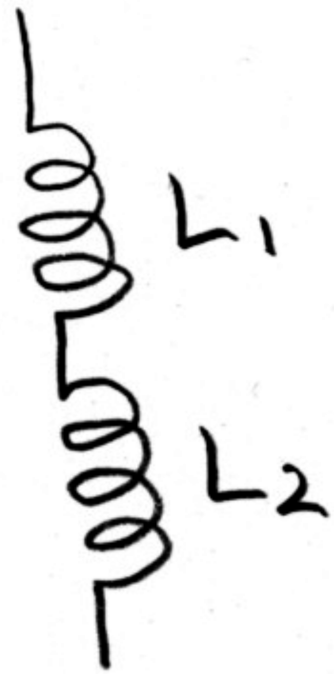
$$I = C\frac{dV}{dt} \qquad V = \frac{1}{C}\int I\,dt$$



$$V = L\frac{dI}{dt} \qquad I = \frac{1}{L}\int V\,dt$$

- Only difference is no magnetic monopole.

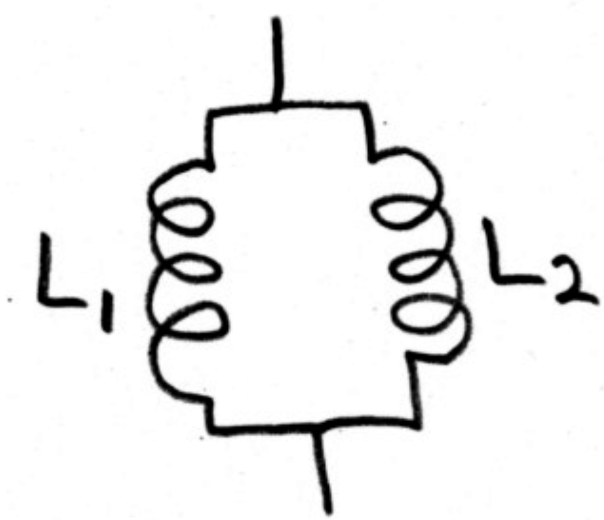
Inductance adds like Resistance



Series

$$L_s = L_1 + L_2$$

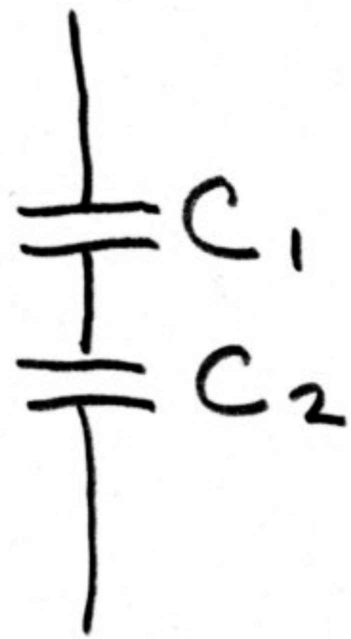
Parallel



$$L_P = \frac{1}{1/L_1 + 1/L_2}$$

$$L_P = \frac{L_1 L_2}{L_1 + L_2}$$

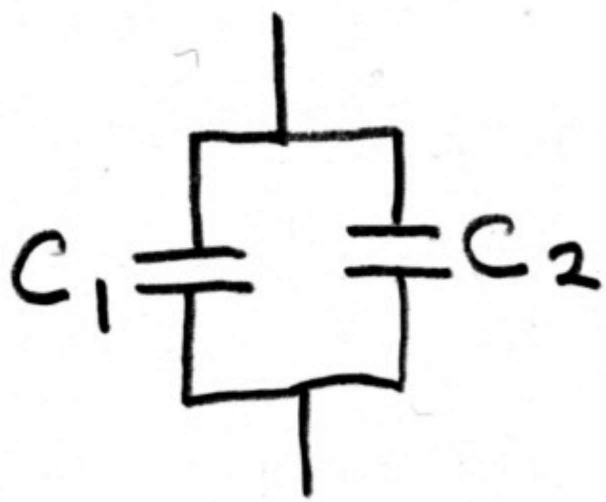
Capacitance adds like *Conductance*



Series

$$C_\mathrm{S} = \frac{1}{1/C_1 + 1/C_2}$$

$$C_\mathrm{S} = \frac{C_1 C_2}{C_1 + C_2}$$



Parallel

$$C_\mathrm{P} = C_1 + C_2$$

Distribution of charge and voltage on multiple capacitors

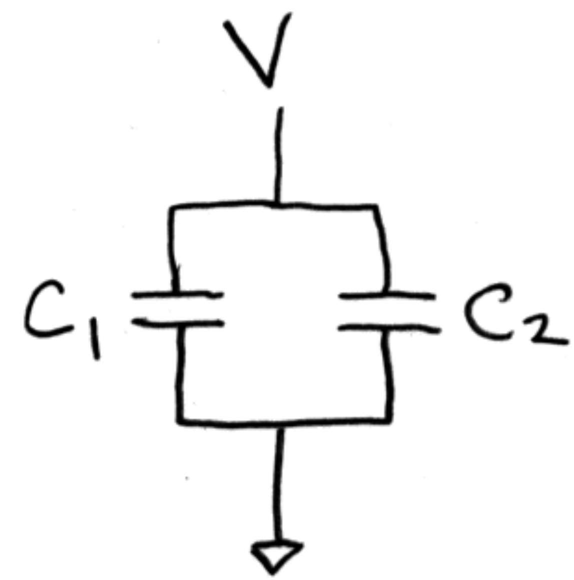
- To find the charge in capacitors in parallel
  - Find total effective capacitance C_{Total}
  - Charge will be $Q_{\text{Total}} = C_{\text{Total}} V$
  - Same voltage will be on all caps (Kirchoff's Voltage Law)

$$Q_{\text{Total}} = VC_{\text{Total}} = Q_1 + Q_2$$

$$V = V_1 = V_2$$

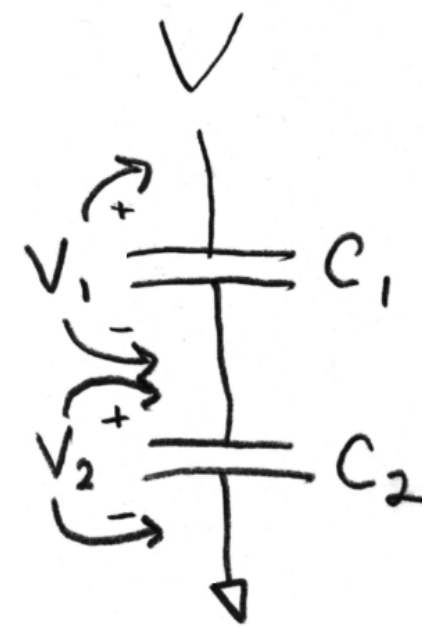
$$Q_1 = VC_1$$

$$Q_2 = VC_2$$



  - Q_{Total} distributed <u>proportional</u> to capacitance

Distribution of charge and voltage on multiple capacitors



- To find the voltages V_1 and V_2 on capacitors in series
  - Find total effective capacitance C_{Total}
  - Charge will be follow the rule for capacitance:

$$Q_{\text{Total}} = C_{\text{Total}} V$$

  - Same charge on both caps (Kirchhoff's Current Law)

$$Q_{\text{Total}} = Q_1 = Q_2$$

($C_{\text{Total}} < C_1$)

V_1 is what portion of V?

$$V_1 = \frac{Q_1}{C_1} = \frac{Q_{\text{Total}}}{C_1} = \frac{C_{\text{Total}}}{C_1} V$$

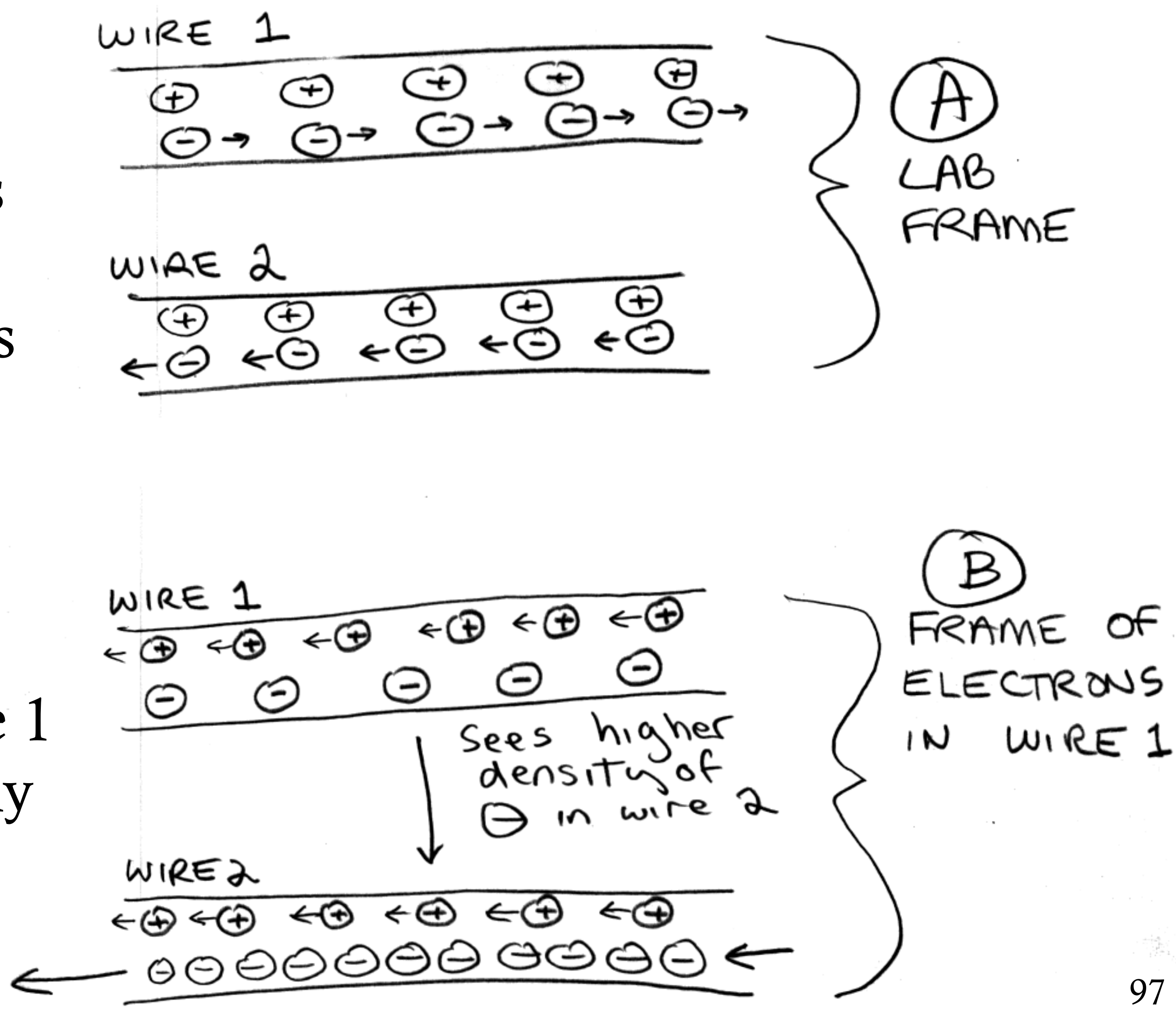
$$V_2 = \frac{Q_2}{C_2} = \frac{Q_{\text{Total}}}{C_2} = \frac{C_{\text{Total}}}{C_2} V$$

  - Voltage distributed <u>inversely proportional</u> to capacitance

What is Magnetism?

- Lorenz Contraction $\ell = \ell_0 \sqrt{1 - v^2/c^2}$

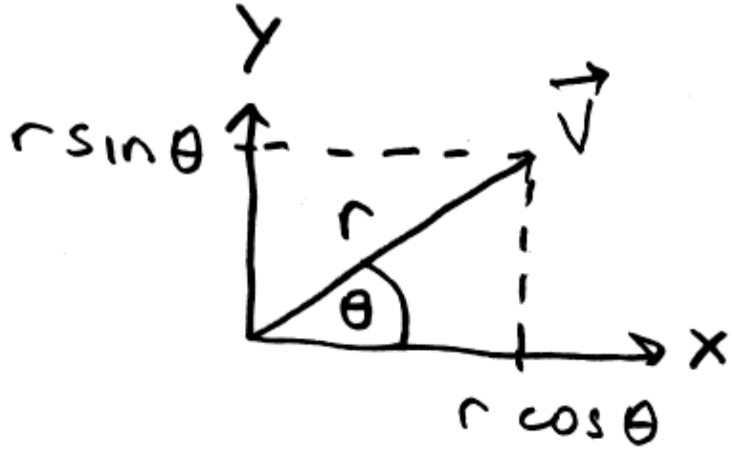
Length ℓ of object observed in relative motion to the object is shorter than the object's length ℓ_0 in its own rest frame as velocity v approaches speed of light c.



Thus electrons in Wire 1 see Wire 2 as negatively charged and repel it: Magnetism!

AC circuit analysis uses Sinusoids

$\cos\theta = \frac{x}{r}$



$\sin\theta = \frac{y}{r}$

saying $\cos^2\theta + \sin^2\theta = 1$ is just the pythagorean theorem

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$x^2 + y^2 = r^2$$

Superposition of Sinusoids

- Adding two sinusoids of the same frequency, no matter what their amplitudes and phases, yields a sinusoid of the same frequency.
- Why? Trigonometry does not have an answer.
- Linear systems change only phase and amplitude
- New frequencies do not appear.

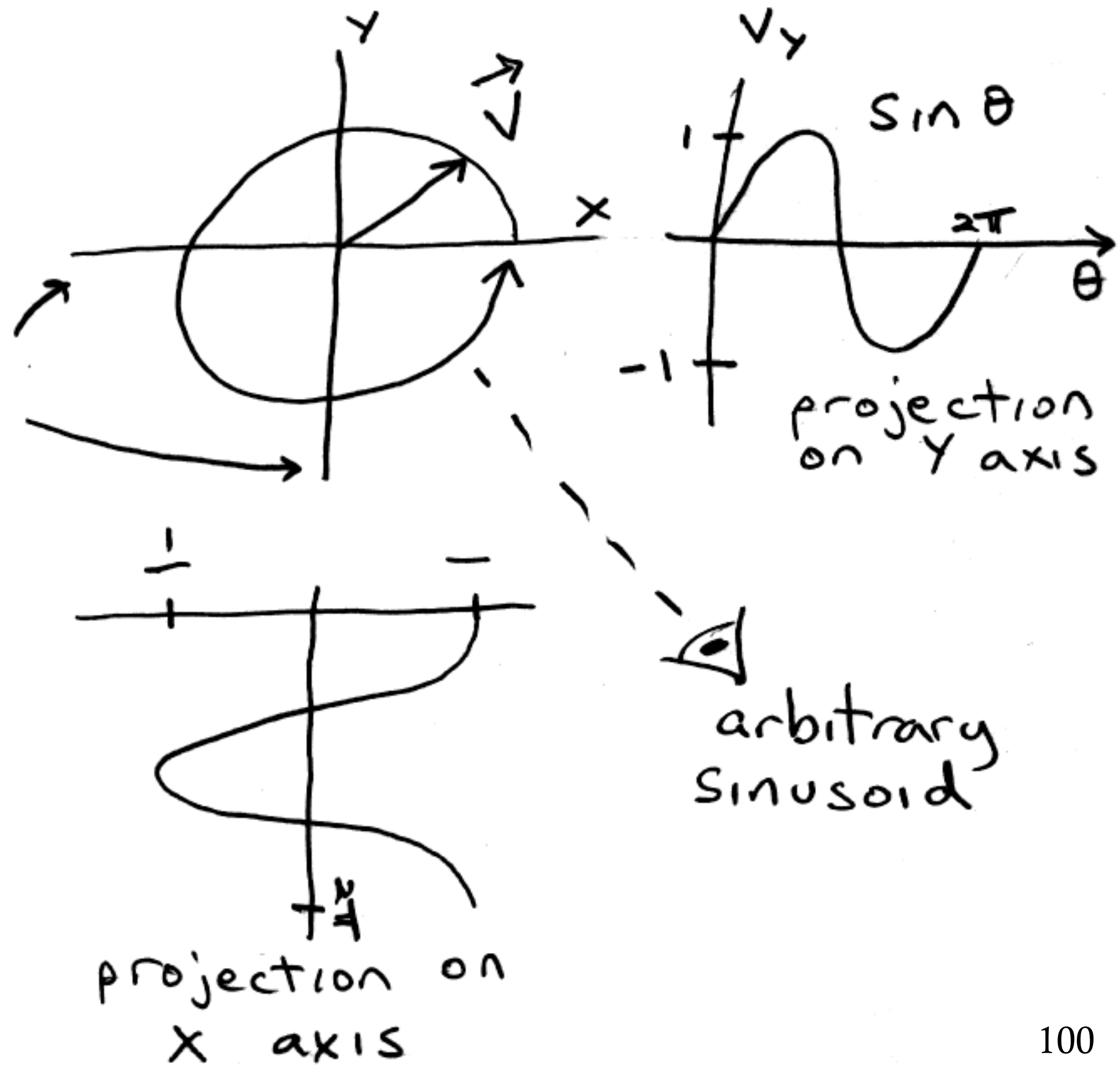
Sinusoids with amplitude of 1 are projections of a unit vector spinning around the origin.

when $r = 1$

$v_x = \cos\theta$

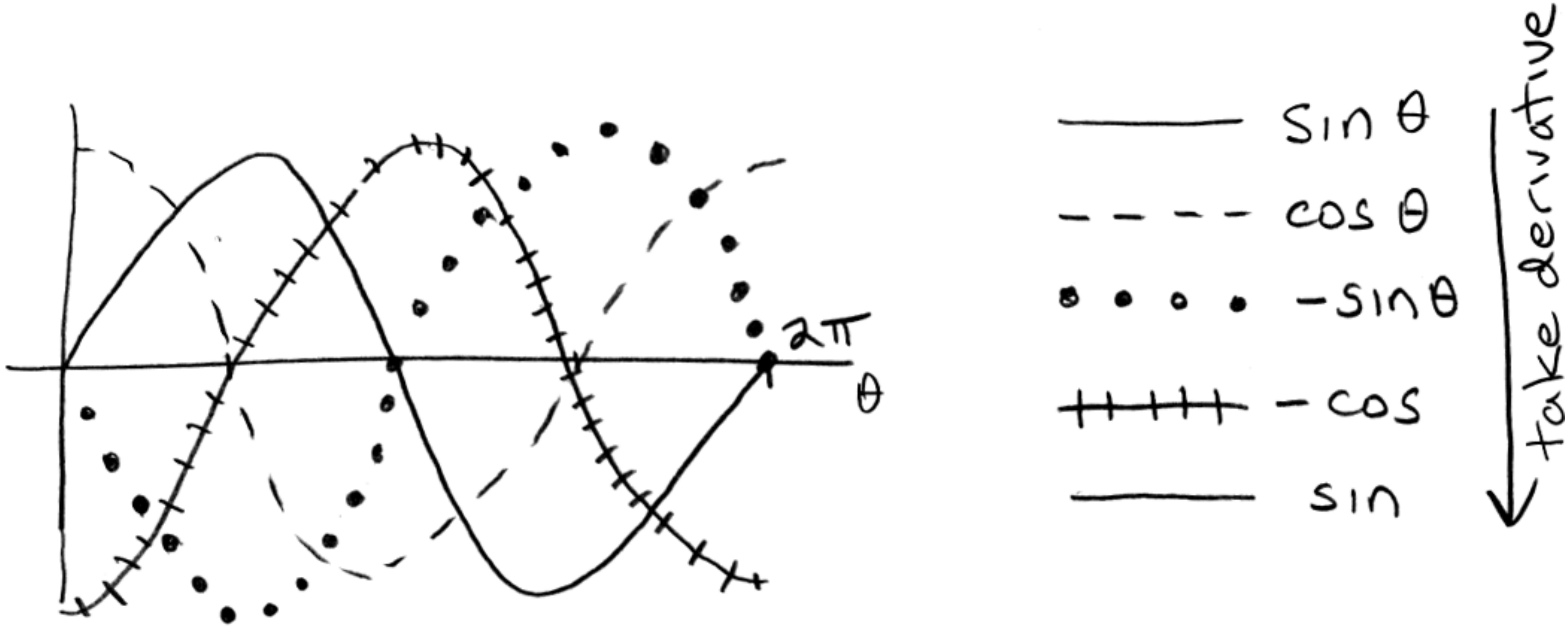
$v_y = \sin\theta$

cardinal axes are just an arbitrary choice



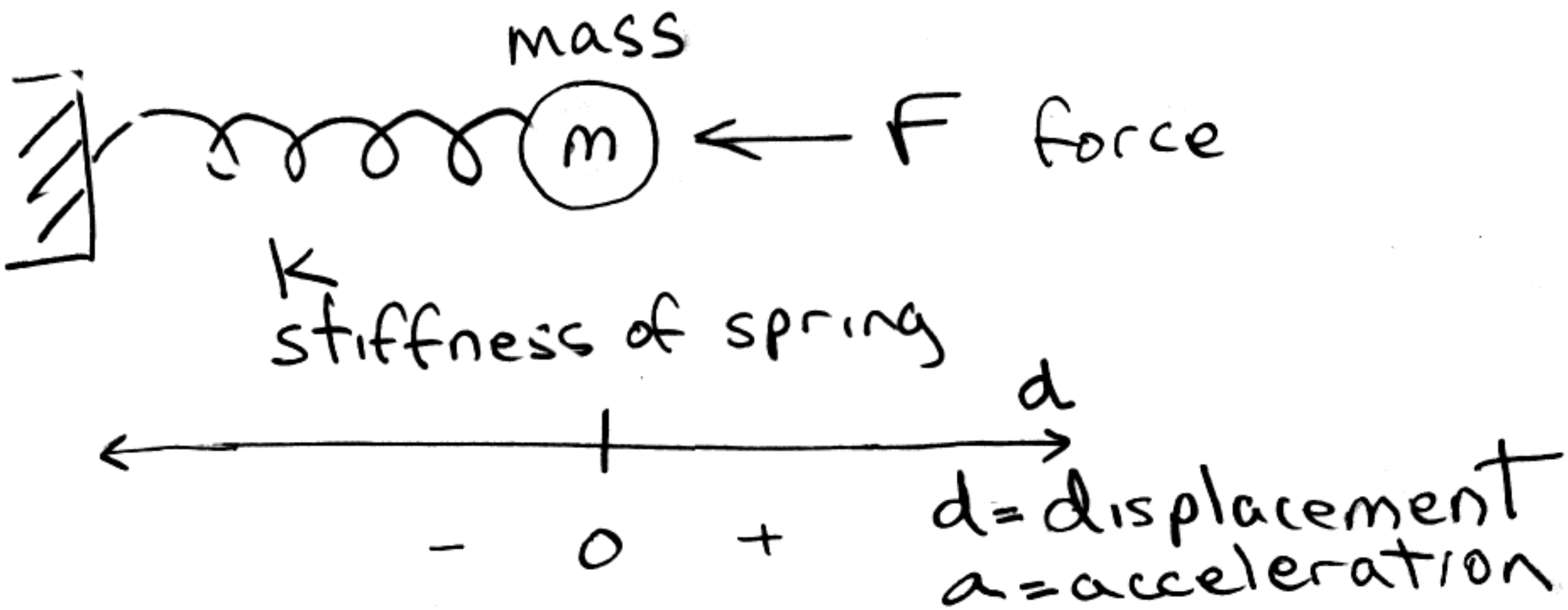
Sin vs cos vs any sinusoid is just a matter of where you say $\theta = 0$

Derivative shifts 90° to the left



Taking a second derivative inverts a sinusoid.

Hooke's Law



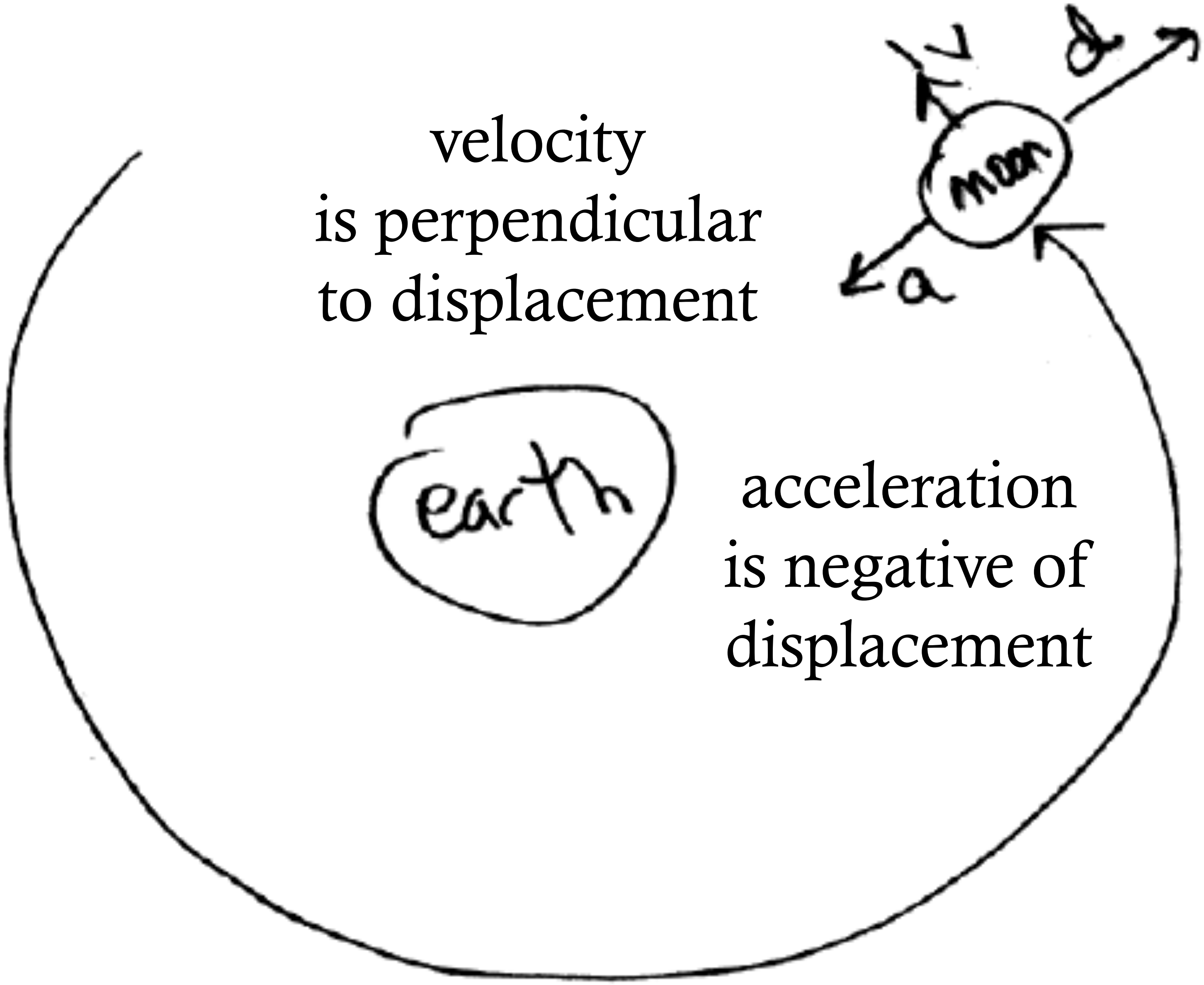
$$F = ma$$
$$F = -kd$$
$$\Rightarrow d = -\left(\frac{m}{k}\right) a$$

constant

Sinusoids result when a function is proportional to its own negative second derivative.

Pervasive in nature: swings, flutes, guitar strings, electron orbits, light waves, sound waves…

Orbit of the Moon – Hook’s Law in 2D



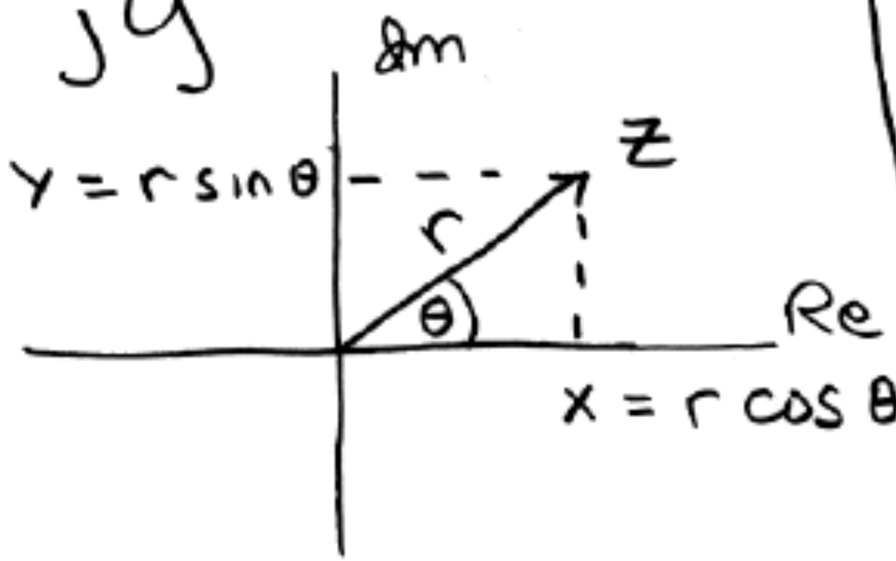
velocity
is perpendicular
to displacement

acceleration
is negative of
displacement

Complex numbers

- Cartesian and Polar forms on complex plane.
- Not vectors, though they add like vectors.
- Can multiply two together (not so with vectors).

$$z = x + jy$$



$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

we say
$-\pi < \theta \leq +\pi$
to make it unique, though θ is actually periodic

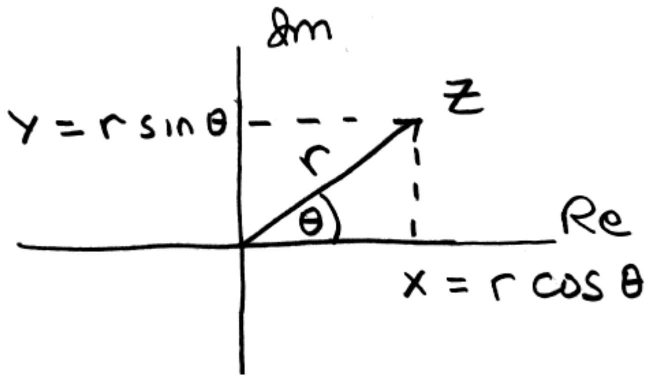
$\theta = \theta + k2\pi$

$k = 0, \pm 1, \pm 2, \dots$

Complex Numbers

- How to find r

$$z = x + jy$$



"modulus" of z, "absolute value" is just a special case where $y=0$.

$$|z| = \sqrt{x^2 + y^2}$$

Note: this is not $\sqrt{z^2}$, but rather the length of the line, r

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$

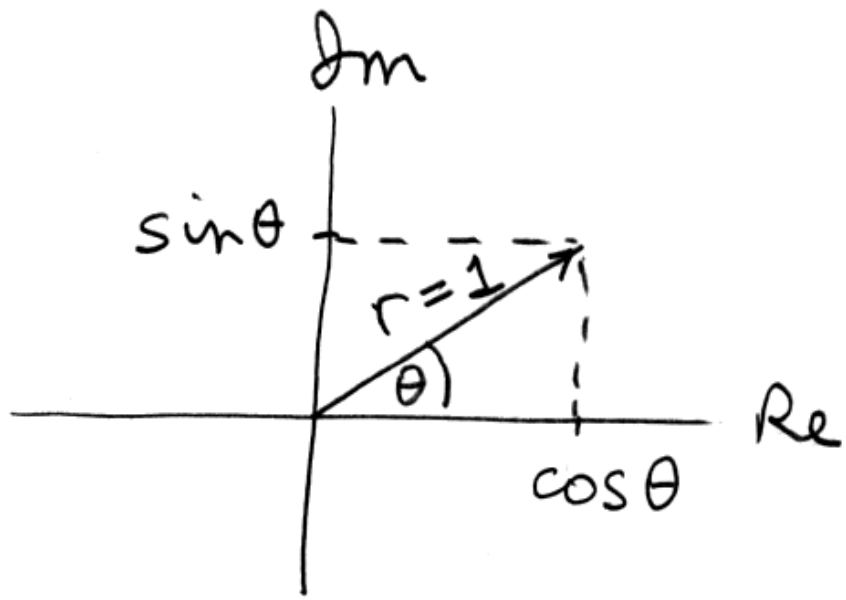
$$= r\sqrt{\underbrace{\cos^2\theta + \sin^2\theta}_{1}}$$

$= r$, which is always ≥ 0

"Phasor" - Polar form of Complex Number

First,

Fixed unit phasor, $r = 1$



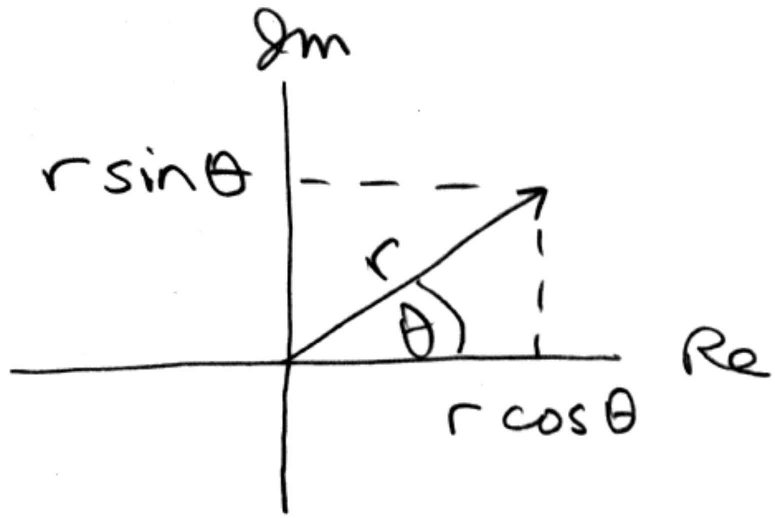
$$e^{j\theta} = \cos\theta + j\sin\theta$$

Euler's Identity

$r = |e^{j\theta}| = 1$ because $\sin^2\theta + \cos^2\theta = 1$

Cartesian and Polar forms (cont…)

Now, for any complex number $z = x + jy$

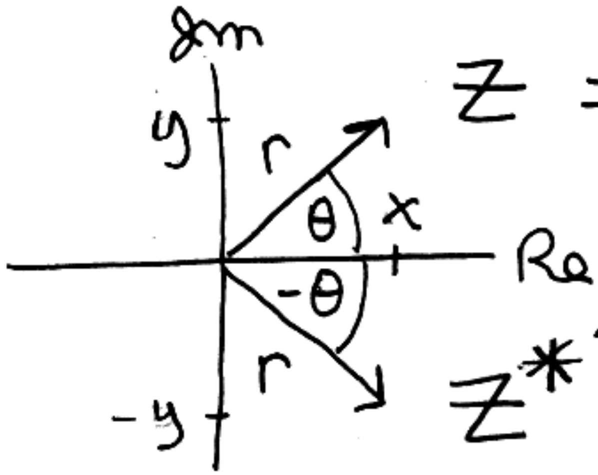


multiply Euler's Identity by r

$$\underbrace{re^{j\theta}}_{\text{polar: } r,\,\theta} = \underbrace{\underbrace{r\cos\theta}_{x} + j\,\underbrace{r\sin\theta}_{y}}_{\text{cartesian: } x,\,y}$$

Complex Conjugates

Complex conjugates – reflect across x-axis



$$z = x + jy = re^{j\theta}$$

asterix means complex conjugate

$$z^* = x - jy = re^{-j\theta}$$

negative sign

product

$$(x+jy)(x-jy) = x^2 + y^2 = r^2$$

or with phasors, phase cancels out

$$(re^{j\theta})(re^{-j\theta}) = r^2 e^{j(\theta-\theta)} = r^2 e^0 = r^2$$

$$z\,z^* = |z|^2$$

"modulus"

Multiplying two complex numbers rotates by each other's phase and scales by each other's magnitude.

messy in Cartesian coordinates → $(x_1 + jy_1)(x_2 + jy_2) =$

$$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) =$$

$$\underbrace{(r_1 r_2)}_{\text{scale each other}} e^{j\overbrace{(\theta_1 + \theta_2)}}$$

rotate each other

Dividing two complex numbers rotates the phase backwards and scales as the quotient of the magnitudes.

even messier $\rightarrow (x_1 + jy_1)/(x_2 + jy_2) =$

$$\underbrace{\frac{r_1}{r_2}}_{\text{scale each other}} e^{j\overbrace{(\theta_1 - \theta_2)}}$$

$$\underbrace{\theta_1 - \theta_2}_{\text{one rotates the other backwards}}$$

How to simplify a complex number in the denominator

$$\frac{1}{Z} = \frac{1}{x+jy} \cdot \frac{x-jy}{x-jy} =$$

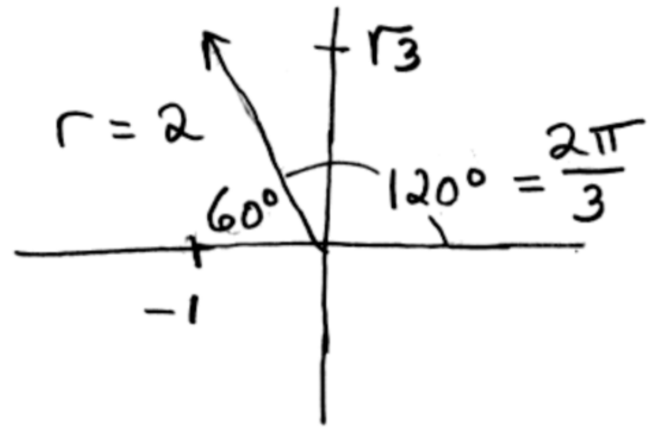
$$\underbrace{\frac{x}{x^2+y^2}}_{\text{real part}} - j\underbrace{\frac{y}{x^2+y^2}}_{\text{imaginary part}} = \frac{Z^*}{|Z|^2}$$

rotate backwards

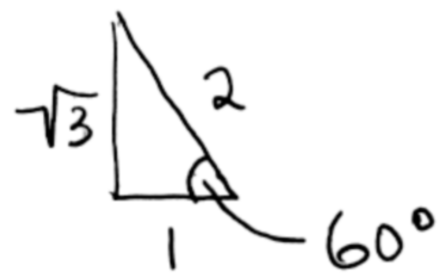
(2) $\boxed{-1+\sqrt{3}\,j}$

$x = -1$

$y = \sqrt{3}$



$$\boxed{2e^{j\frac{2\pi}{3}}}$$



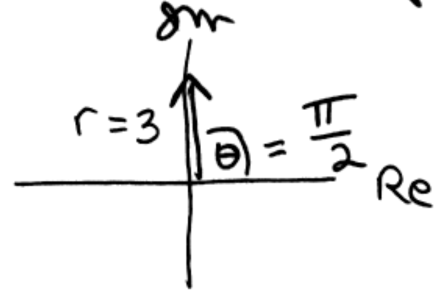
in general

$$re^{j\theta} = \sqrt{x^2+y^2}\, e^{j \arctan\left(\frac{y}{x}\right)}$$

Going the other way

Convert the following complex numbers to cartesian coordinates $x + jy$ drawing a picture in the complex plane

① $3e^{j\frac{\pi}{2}}$



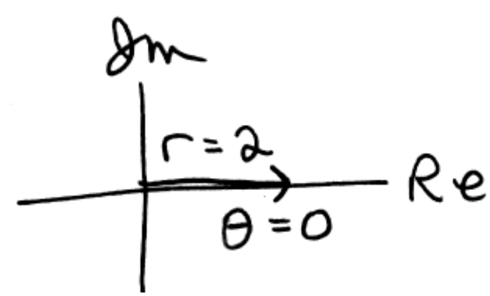
$x = 0 \;,\; y = 3$

$z = 0 + j3$

---

② $-2e^{-j3\pi} = \underbrace{-2e^{-j\pi}}_{\substack{\text{since} \\ e^{j\theta} = e^{j(\theta + 2\pi K)} \\ K = 0, \pm 1, \pm 2, \ldots}} = \underbrace{2e^{0}}_{\substack{\text{since} \\ e^{j\theta} = -e^{j(\theta \pm \pi)}}} = 2 + j0$

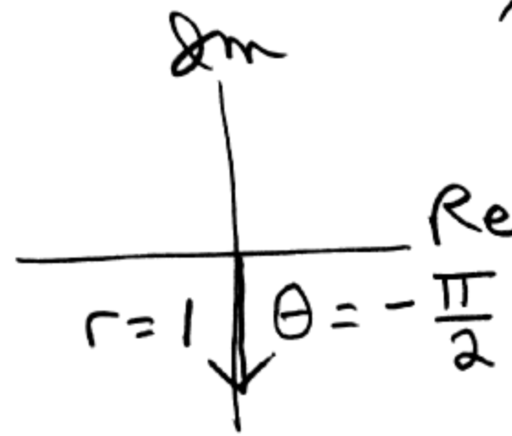
$x = 2$
$y = 0$



③ $je^{j\pi} = e^{j\frac{\pi}{2}}e^{j\pi} = e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}} = 0 - j$

$x = 0$

$y = -1$



In general

$x = \mathrm{Re}\{re^{j\theta}\}$ $\quad$ $y = \mathrm{Im}\{re^{j\theta}\}$

The "squiggly" bracket: not an algebraic expression.

y itself is real: the coordinate on the imaginary axis

Examples

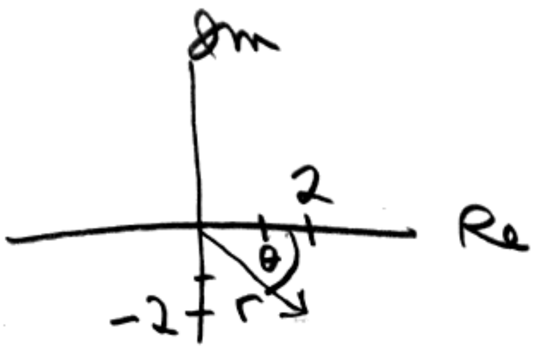
convert the following complex numbers to polar coordinates $re^{j\theta}$

$r \geq 0 \quad -\pi < \theta \leq \pi$

① $\boxed{2-2j}$

$x = 2$

$y = -2$



$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\theta = -45° = -\frac{\pi}{4}$

therefore, $2-2j = \boxed{2\sqrt{2}\, e^{-j\frac{\pi}{4}}}$

Let's review the dimensionality of phase and frequency...

θ = phase = angle

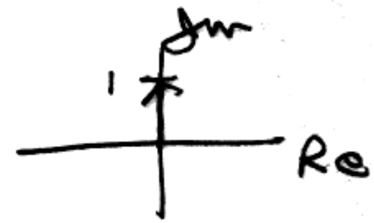
usually in radians, but can be degrees, or cycles

1 cycle = 360 degrees = 2π radians

$e^{j(\;)}$ this is phase

Rotating any complex number by + or - 90°

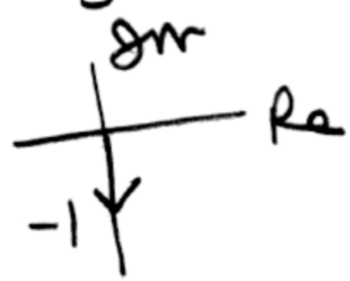
$$j = \underbrace{1}_{r}\, e^{j\underbrace{\left(\frac{\pi}{2}\right)}_{\theta}}$$



Multiplying by j
rotates any complex number
by 90°

dividing by j rotates by −90°

because $\frac{1}{j} = \frac{1}{j}\,\frac{-j}{-j} = -j = 1e^{-j\frac{\pi}{2}}$



phase = frequency × time, as in

$$e^{j(\omega t)} = e^{j(2\pi f t)}$$

$$\omega = 2\pi f$$

frequency in radians/sec

frequency in cycles/sec

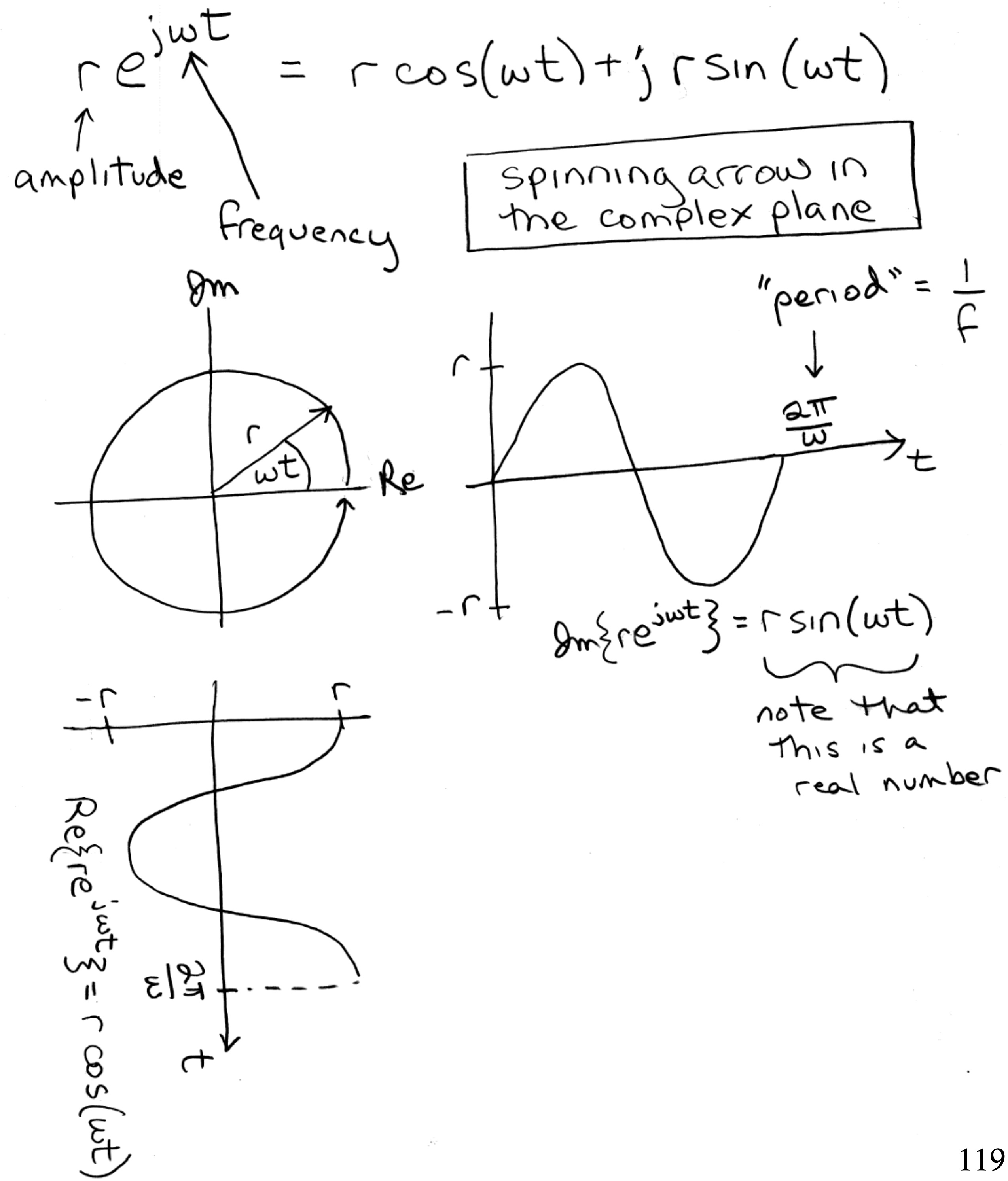
$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

<u>period</u>, seconds/cycle

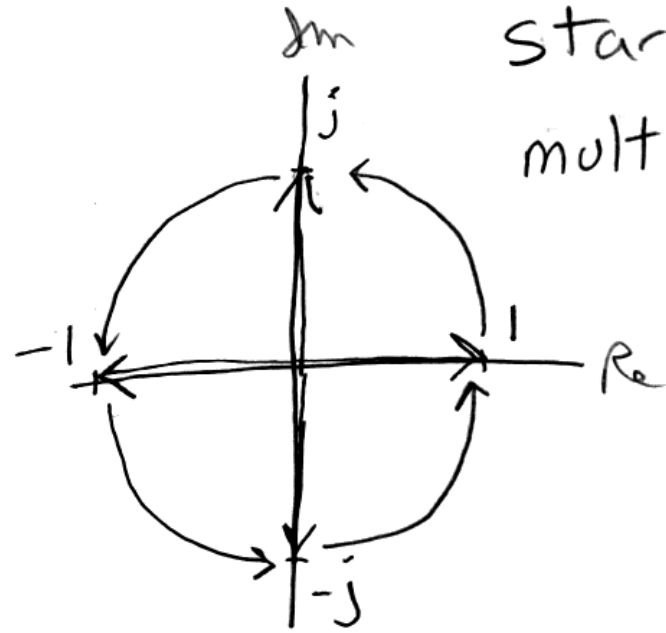
2π, not π,
is the magic number

Now make the phasor spin at $\omega = 2\pi f$

Note: frequency can be negative; phasor can spin backwards.



Multiplying by *j* shifts the phase by 90°



start with 1

multiply by j, you get j, -1, -j, 1

shift by $+90° = +\frac{\pi}{2}$

phasors do the same thing when you take their derivative:

$$\frac{de^{jt}}{dt} = j e^{jt} \qquad (\omega = 1)$$

$$\frac{d^2 e^{jt}}{dt^2} = j \cdot j e^{jt} = -e^{jt}$$

solution to Hooke's Law

$$j = e^{j\frac{\pi}{2}}$$

$\theta = 90°$

Just like a sinusoid: shifts 90° with each derivative.

- All algebraic operations work with complex numbers
- What does it mean to raise something to an imaginary power?
- Consider case of $e^{j\omega t}$ with $\omega = 1$

$$e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \cdots$$

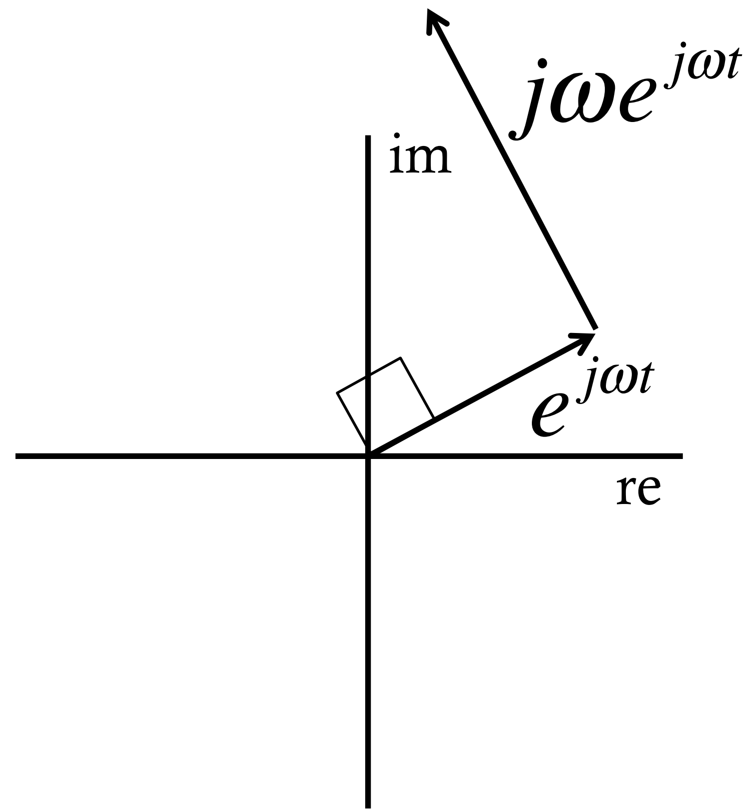
$$\sin(t) = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} \cdots$$

$$\left.\begin{aligned} \cos(t) &= 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 \cdots \\ j\sin(t) &= 0 + jt + 0 - j\frac{t^3}{3!} + 0 + j\frac{t^5}{5!} \cdots \\ \hline e^{jt} &= 1 + jt - \frac{t^2}{2} - j\frac{t^3}{3!} + \frac{t^4}{4!} + j\frac{t^5}{5!} \cdots \end{aligned}\right\}$$

$$\boxed{e^{jt} = \cos(t) + j\sin(t)}$$

<u>Euler's Identity</u>

Consider $e^{j\omega t}$ graphically.



Its derivative

$$\frac{de^{j\omega t}}{dt} = j\omega e^{j\omega t}$$

is rotated by 90° and scaled by ω at all times.
Thus it spins in a circle with velocity ω,
and since $e^{j\omega t} = 1$ when $t = 0$,

$$\boxed{e^{j\omega t} = \cos\omega t + j\sin\omega t}$$

Euler's Identity

Voltages and Currents are Real

$$\mathrm{Re}\{z\} = x = \frac{(x+jy)+(x-jy)}{2} = \frac{z+z^*}{2}$$

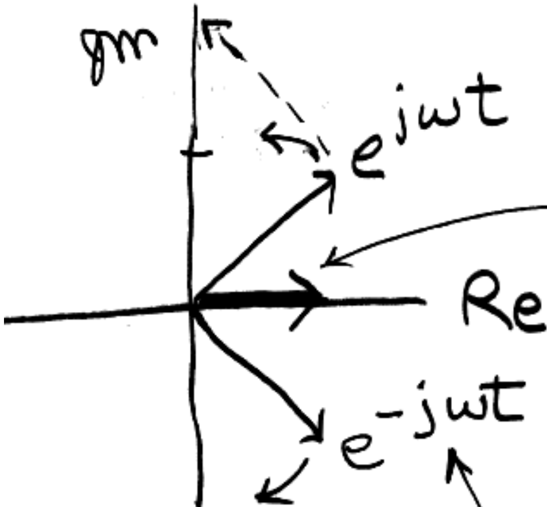
$$\mathrm{Im}\{z\} = y = \frac{(x+jy)-(x-jy)}{2j} = \frac{z-z^*}{2j}$$

The imaginary coordinate y is itself real

$$z = x + jy$$

this is imaginary.

Cosine is sum of 2 phasors



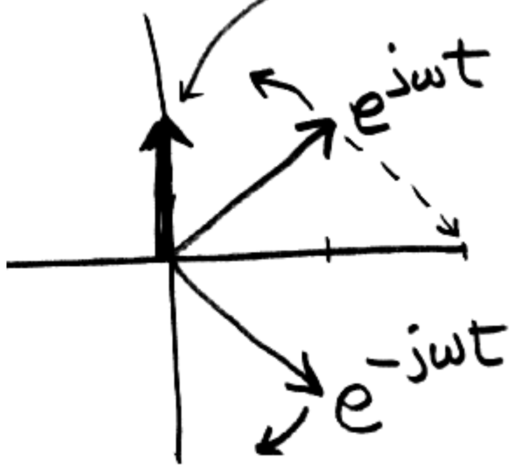
complex conjugates

$$\cos(\omega t) = \mathrm{Re}\{e^{j\omega t}\} = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

to get cos:
subtract off
imaginary part

negative frequency,
backwards spinning phasor, complex conjugate

Sine is difference between 2 phasors



$$\sin(wt) = \underbrace{\mathrm{Im}\{e^{jwt}\}}_{\substack{\text{this is a}\\ \text{real number}\\ \text{so must divide}\\ \text{by } j}} = \frac{e^{jwt} - e^{-jwt}}{2j}$$

$$\boxed{\text{to get sin: subtract off real part.}}$$

Trigonometry Revealed

remember the end of trig?

$$\sin 2t = 2 \sin t \cos t$$

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$\sin \frac{1}{2} t = \pm \sqrt{\frac{1}{2}(1 - \cos t)}$$

you had to take them
on faith...
No longer!

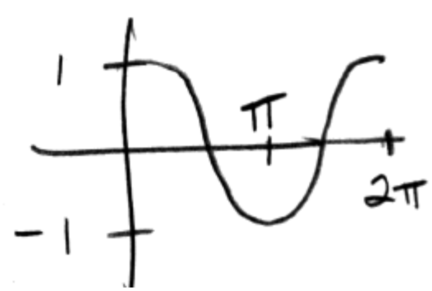
Now you can <u>prove</u> them

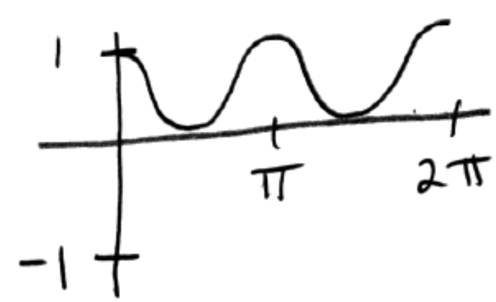
example

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right)^2 = \frac{e^{j2\theta}+e^{-j2\theta}+e^0+e^0}{4}$$

$$= \frac{\cos 2\theta}{2} + \frac{1}{2}$$

squaring 

yields 

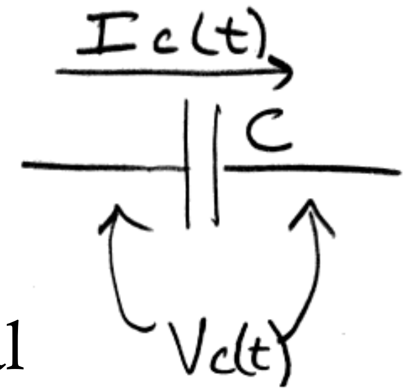
Why have we learned the math of phasors?

- We will now see how resistance is just the real part of a complex parameter, *impedance.*
- Resistors have real impedance. Capacitors and inductors have imaginary impedance.
- All the laws we have learned in DC for resistance apply in AC for impedance.
- Thus we can solve complicated differential equations using algebra (of complex numbers).
- To derive impedance, we consider the function $e^{j\omega t}$ as the *orthogonal basis set* from which any voltage or current can be built (Fourier).

Complex Impedance - Capacitor

Complex impedance Z replaces resistance R (which is real)

$$I_c(t) = C\,\frac{dV_c(t)}{dt}$$



if $V_c(t) = e^{j\omega t}$ ← represents orthogonal basis set

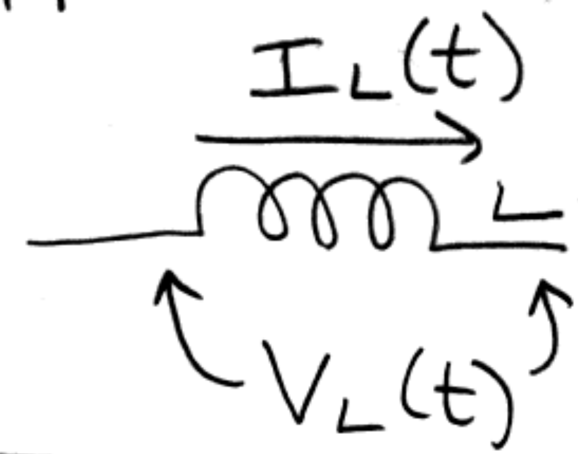
then $I_c(t) = j\omega C e^{j\omega t}$

Complex Impedance using Ohms Law

$$Z_c = \frac{V_c(t)}{I_c(t)} = \frac{e^{j\omega t}}{j\omega C e^{j\omega t}} = \boxed{\frac{1}{j\omega C}} = -\frac{j}{\omega C}$$

Complex Impedance - Inductor

Likewise for a coil



$$V_L(t) = L\,\frac{dI_L(t)}{dt}$$

if $I_L(t) = e^{j\omega t}$

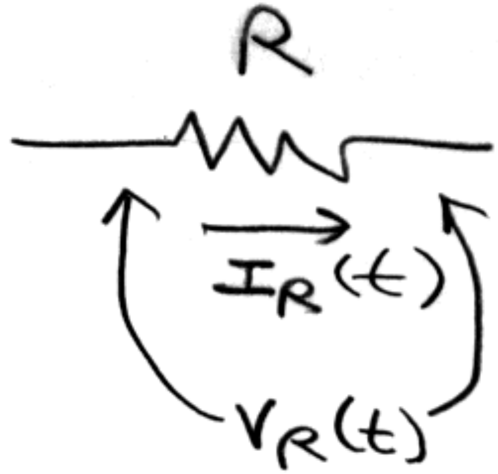
then $V_L(t) = j\omega L\, e^{j\omega t}$

Complex impedance

$$Z_L = \frac{V_L(t)}{I_L(t)} = \frac{j\omega L e^{j\omega t}}{e^{j\omega t}} = \boxed{j\omega L}$$

Complex Impedance - Resistor

what is complex impedance of resistor



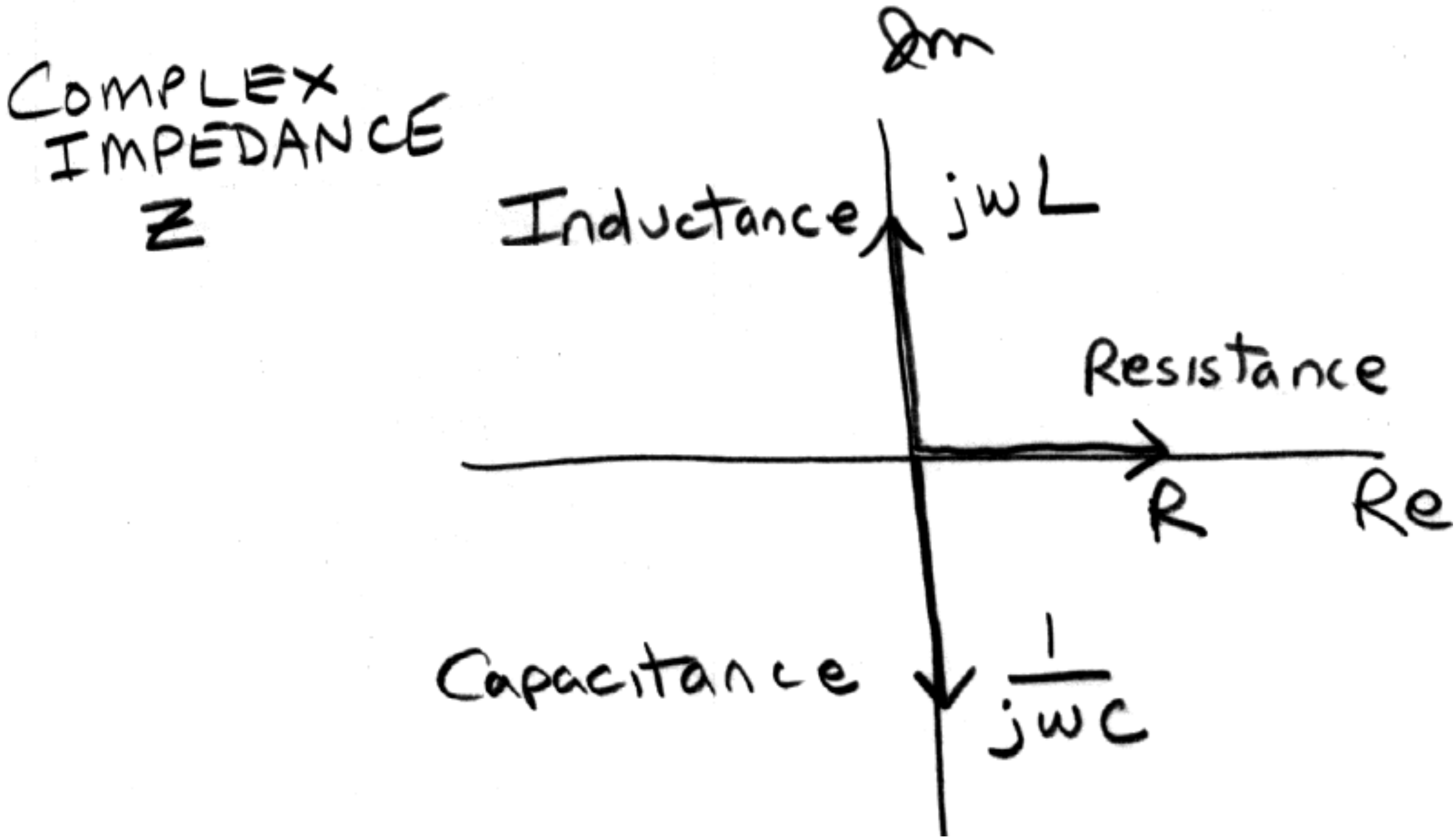
$$V_R(t) = e^{j\omega t}$$

$$I_R(t) = \frac{e^{j\omega t}}{R}$$

$$Z_R = \boxed{R}$$

pure <u>real</u>

Impedance on the Complex Plane



Taxonomy of Impedance

Complex Impedance

impedance Z $\xrightarrow{\text{real}}$ resistance $R = \dfrac{1}{\text{conductance } G}$

impedance Z $\xrightarrow{\text{imaginary}}$ reactance X

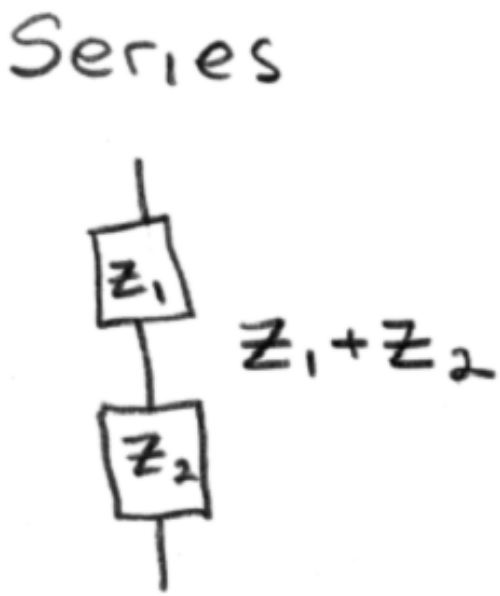
$\dfrac{1}{Z}$ = "admittance"

reactance X $\rightarrow$ capacitance $X_C = \dfrac{1}{j\omega C}$

reactance X $\rightarrow$ inductance $X_L = j\omega L$

"X" sometimes used if purely imaginary, or just "Z_C" and "Z_L"

Series Capacitors and Inductors



Two capacitors in series:

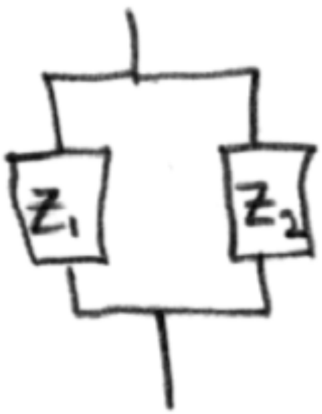
$$Z = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} = \frac{C_1 + C_2}{j\omega C_1 C_2} = \frac{1}{j\omega\left(\dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}\right)}$$

Two inductors in series:

$$Z = j\omega L_1 + j\omega L_2 = j\omega\left(L_1 + L_2\right)$$

Parallel Capacitors and Inductors

Parallel



$$\frac{1}{\frac{1}{Z_1}+\frac{1}{Z_2}}$$

Two capacitors in parallel:

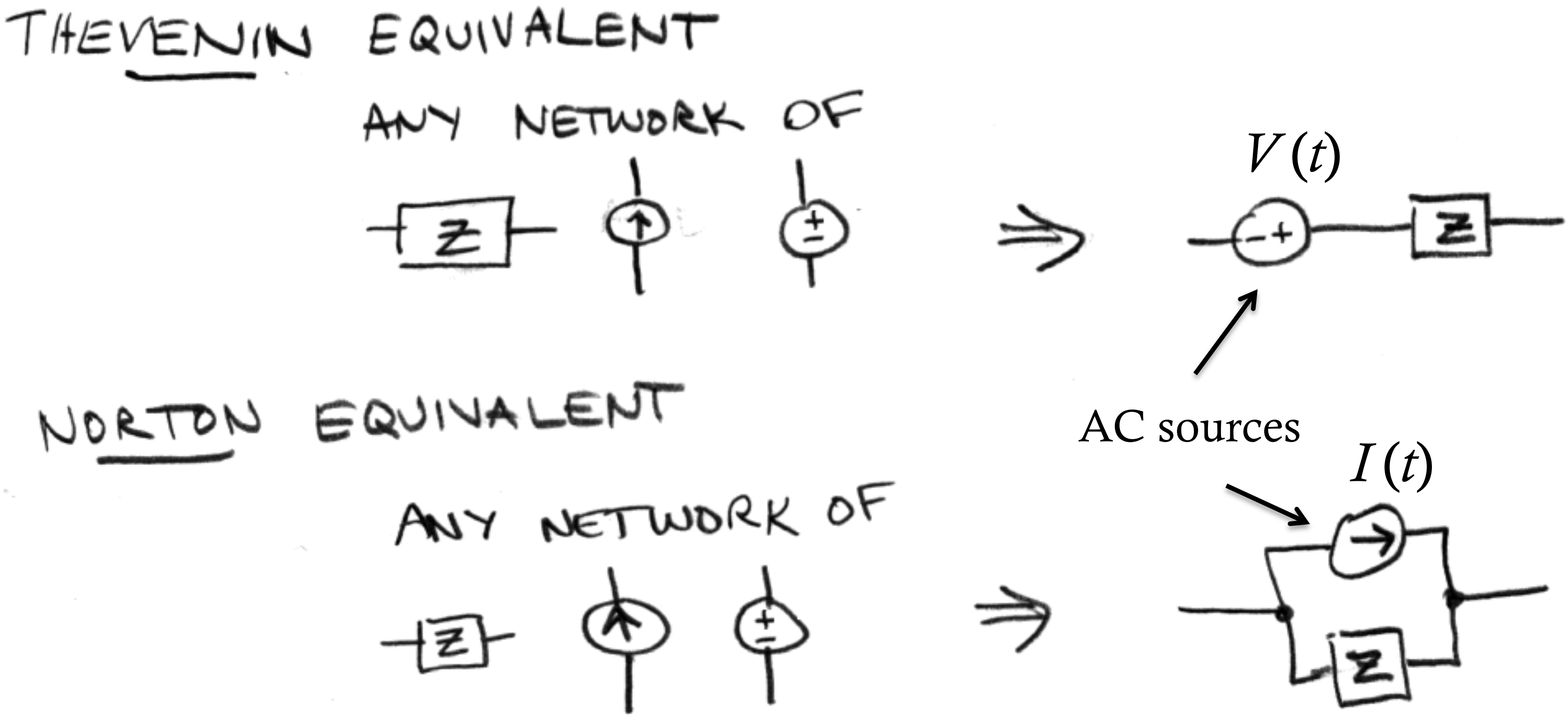
$$Z = \frac{1}{j\omega C_1 + j\omega C_2} = \frac{1}{j\omega\left(C_1 + C_2\right)}$$

Two inductors in parallel:

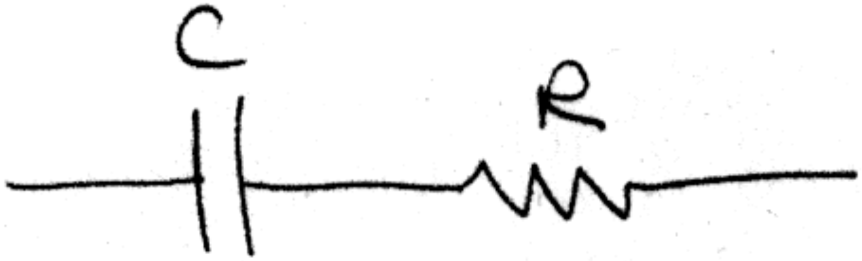
$$Z = \frac{1}{\frac{1}{j\omega L_1} + \frac{1}{j\omega L_2}} = j\omega\left(\frac{1}{\frac{1}{L_1} + \frac{1}{L_2}}\right)$$

Same rules as DC circuits

Now using AC voltage and current sources and complex impedance Z



Impedance of a Passive Branch – RC circuit

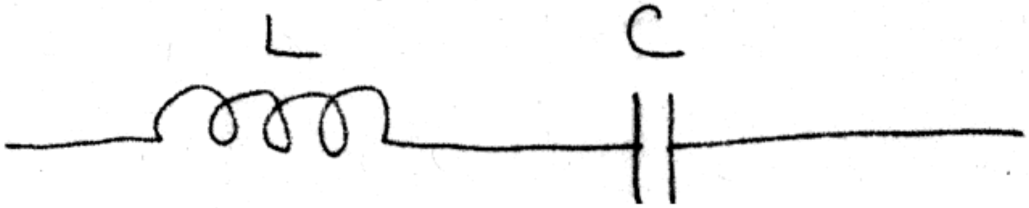


$$Z = \frac{1}{j\omega C} + R = \frac{1 + j\omega \overbrace{RC}^{\text{time constant}}}{j\omega C}$$

$$Z = R \,\Big|_{\omega \gg \frac{1}{RC}} \quad \text{Resistor dominates}$$

$$Z = \frac{1}{j\omega C} \,\Big|_{\omega \ll \frac{1}{RC}} \quad \text{Capacitor dominates}$$

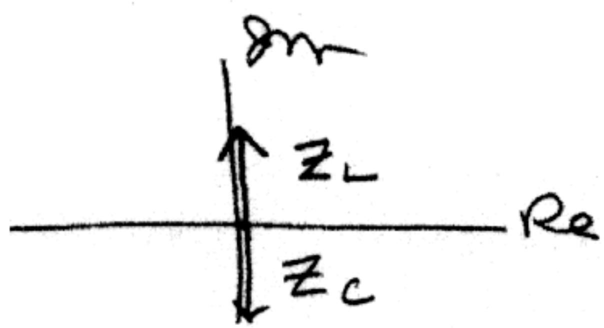
LC circuit - Resonance



$$Z = j\omega L + \frac{1}{j\omega C} = \frac{1-\omega^2 \overbrace{LC}^{\text{time constant}}}{j\omega C}$$

$$Z = 0 \Big|_{\omega = \frac{1}{\sqrt{LC}}} \quad \text{Resonance}$$

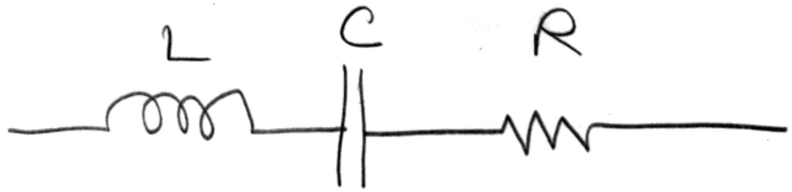
At resonance, impedances add to zero and cancel.



pure "reactance"

Analogous to spring and weight system – Energy in passed between magnetic and electric fields, as in electromagnetic wave.

Adding R to LC damps the ringing

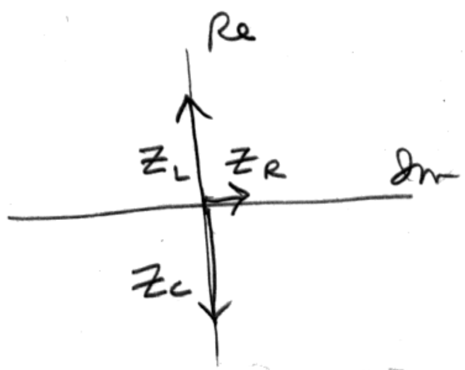


$$Z = j\omega L + \frac{1}{j\omega C} + R =$$

$$\frac{1-\omega^2 LC}{j\omega C} + R =$$

$$\underbrace{j\left(\frac{\omega^2 LC - 1}{\omega C}\right)}_{\text{pure imaginary}} + R$$

pure Real

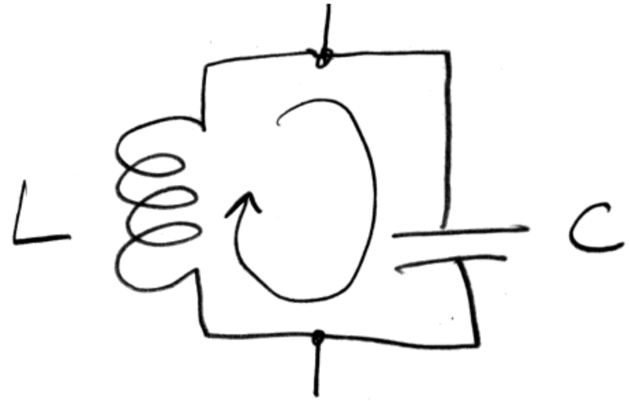


can never equal zero

Like dragging your feet on the swing. Energy being passed from magnetic to electric field eventually dissipated by resistor as heat.

“Tank” Circuit

put coil and cap in loop



COOL FACT
speed of light

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

"inductance" and "capacitance" of free space

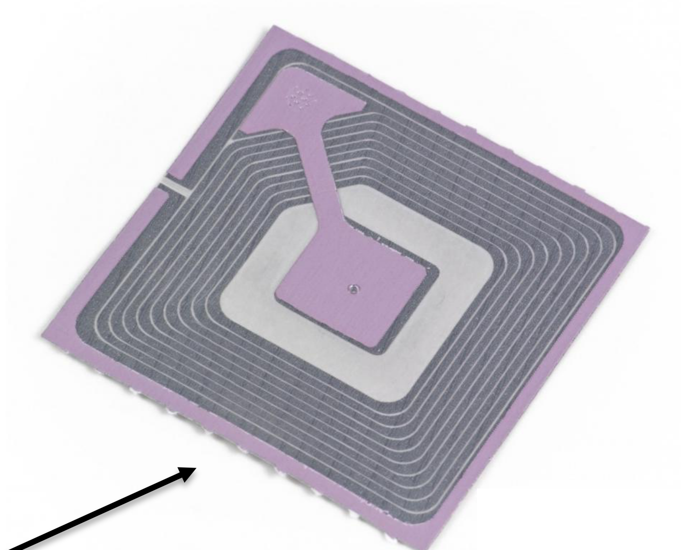
permiability and permitivity

current *around* loop sees no impedance

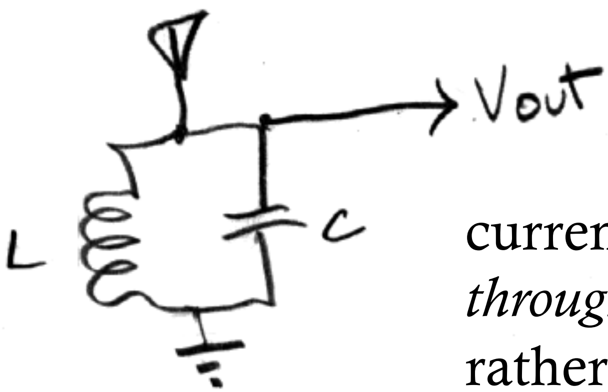
at $\omega = \frac{1}{\sqrt{LC}}$

will resonate and absorb radio frequency (RF) energy

used in anti shoplifting tags



"Tank" circuit first thing after antenna in radio receiver



$$Z = \frac{1}{\frac{1}{j\omega L} + \frac{1}{\frac{1}{j\omega C}}} = \frac{1}{\frac{1}{j\omega L} + j\omega C} = \frac{j\omega L}{1 - \omega^2 LC} = \infty \Big|_{\omega = \frac{1}{\sqrt{LC}}}$$

current is now *through* loop rather than *around* it

- How can impedance be infinite through the parallel LC circuit when each of the components can pass current?
- At the resonant frequency the currents trying to pass from the antenna to ground are shifted 90° in opposite directions and thus are 180° out of phase and cancel. No net current!
- This "null point" is an example of destructive interference, how lenses work with light (described by phasors 3D space).

Phasor Notation

- In BioE 1310, complex exponentials may be described with shorthand "phasor notation"

$$re^{j\theta} \Rightarrow "r\angle\theta"$$

- Unfortunately, this abbreviation is widely used to represent *real* voltages and currents, with no consensus as to whether it means sin or cos, peak or root mean squared (RMS). Thus, $A\angle\theta$ may mean (among other things)

$$v(t) = \frac{A}{\sqrt{2}}\sin(\omega t + \theta)$$

or

$$v(t) = A\cos(\omega t + \theta)$$

Phasor Notation Ambiguity (cont…)

- This ambiguity is allowed to continue because linear systems change only magnitude and phase.
- Thus a given network of coils, capacitors, and resistors will cause the same relative change in

$$v(t)=\frac{A}{\sqrt{2}}\sin(\omega t+\theta)$$

as it does in

$$v(t)=A\cos(\omega t+\theta)$$

so it doesn't matter which definition of $A\angle\theta$ is used for real signals, so long as it remains consistent.

http://www.vialab.org/Bioe_1310/pdf/PhasorNotationManifesto.pdf

Sample Problems with Phasor Notation

Using our unambiguous definition of phasor notation,

$$r\angle\theta = re^{j\theta}$$

Express the following as a complex number in Cartesian form (x + jy):

$$(4\angle 45°)(6\angle 45°) = 24\angle 90° = 0 + 24j$$

- In other words, for multiplication, multiply the magnitudes and add the phases.
- For division, divide the magnitudes and subtract the phases.

$$\frac{6\angle 30°}{3\angle 90°} = 2\angle -60° = 1 - j\sqrt{3}$$

Another look at Superposition

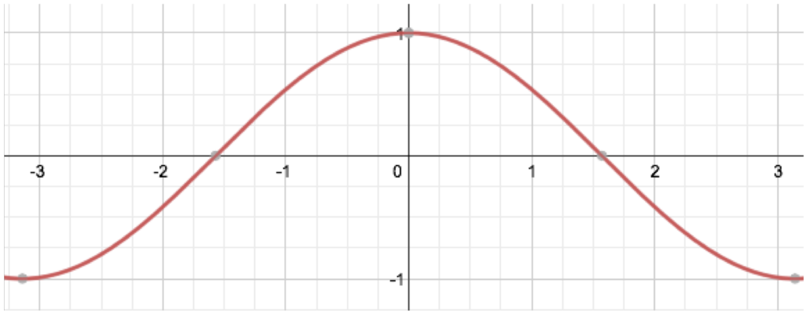
$$A\sin(\omega t)+B\cos(\omega t)$$

combine to form a sinusoid with frequency ω,

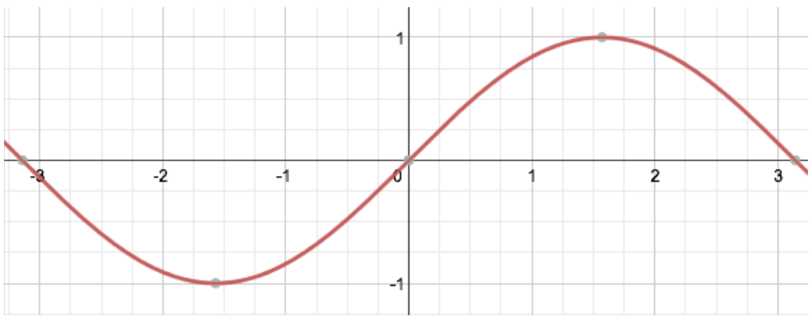
and how any sum of sinusoids with frequency ω

$$\sum_i \underbrace{\overset{\text{amplitude}}{A_i}\cos\left(\overset{\text{frequency}}{\omega} t+\overset{\text{phase}}{\theta_i}\right)}_{\text{any sinusoid of frequency }\omega} \text{ is a sinusoid with frequency } \omega$$

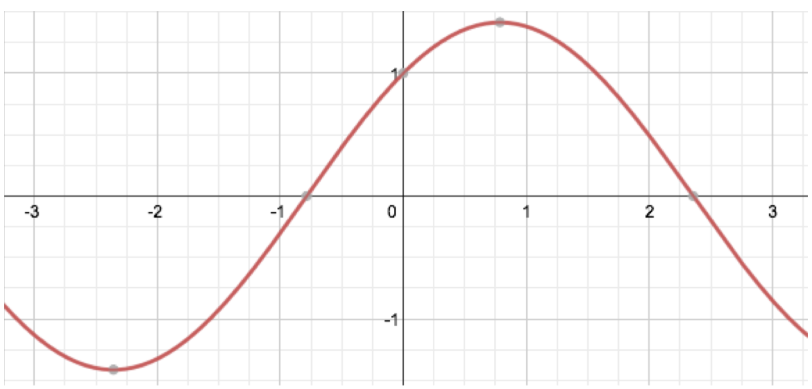
Example: $\cos(t) + \sin(t), \quad \omega = 1$



$$\cos(t)$$



$$\sin(t)$$

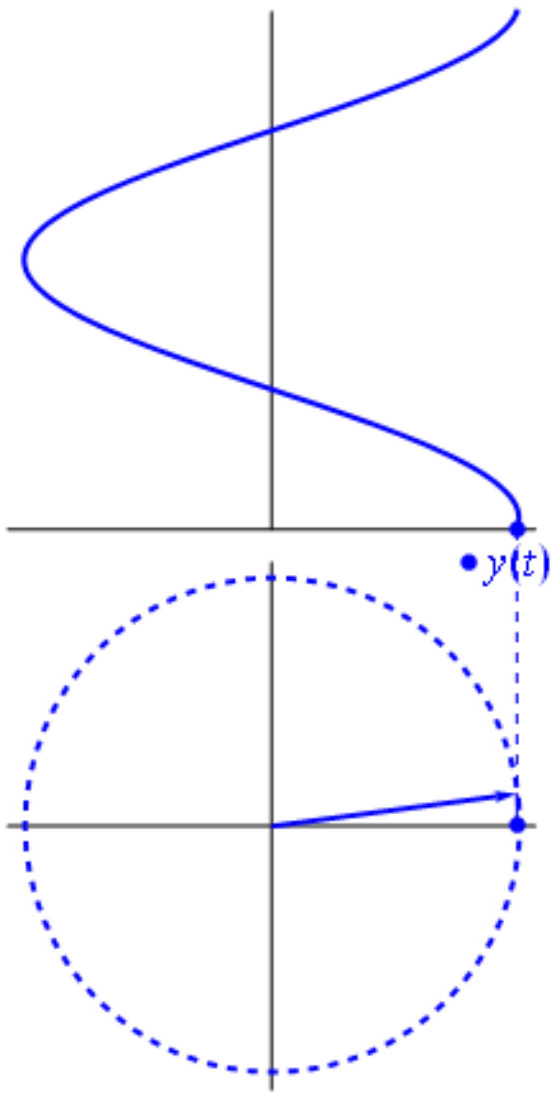


$$\cos(t) + \sin(t) =$$

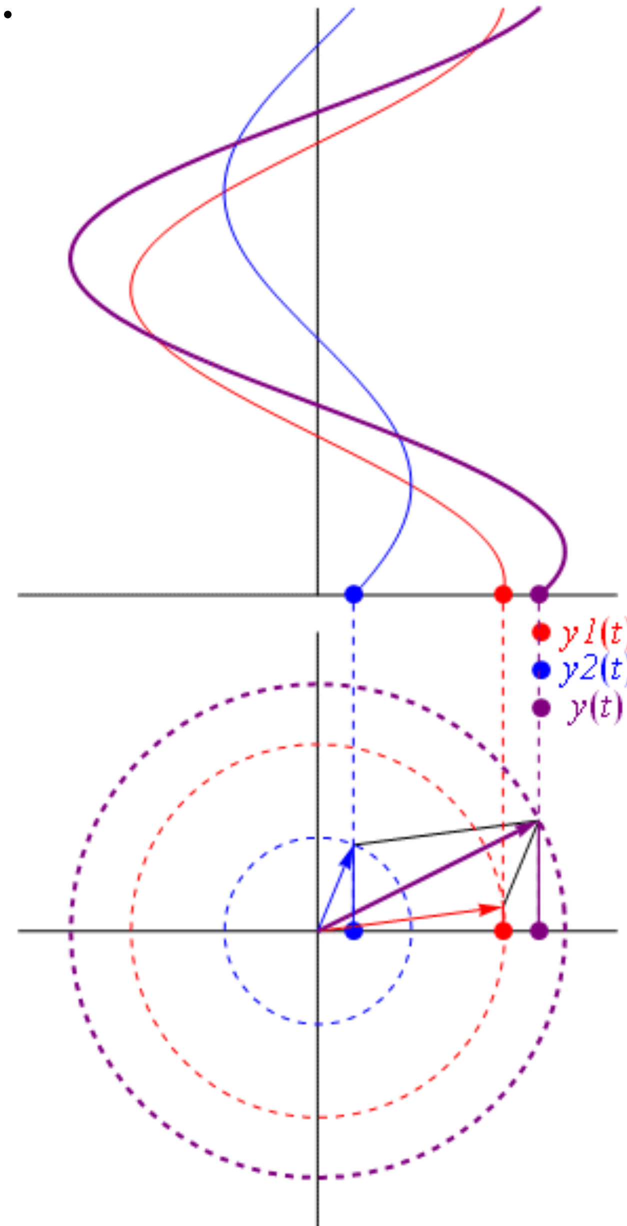
$$\sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$$

… is sinusoid of same frequency.

- Two phasors of the same frequency and direction sum to a third phasor of the same frequency and direction.
- They form a rigid spinning body.



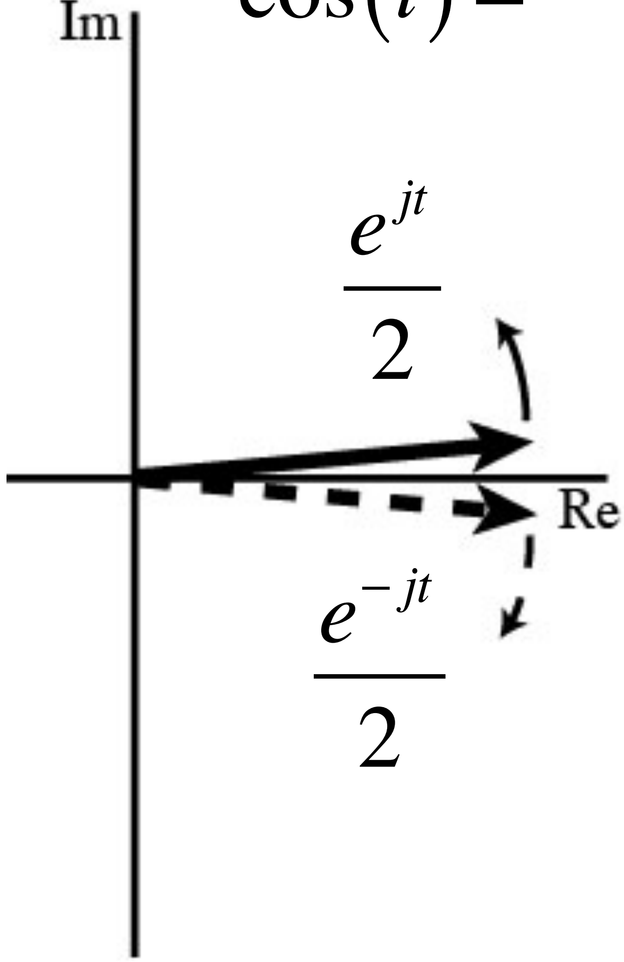
Single phasor $y(t)$



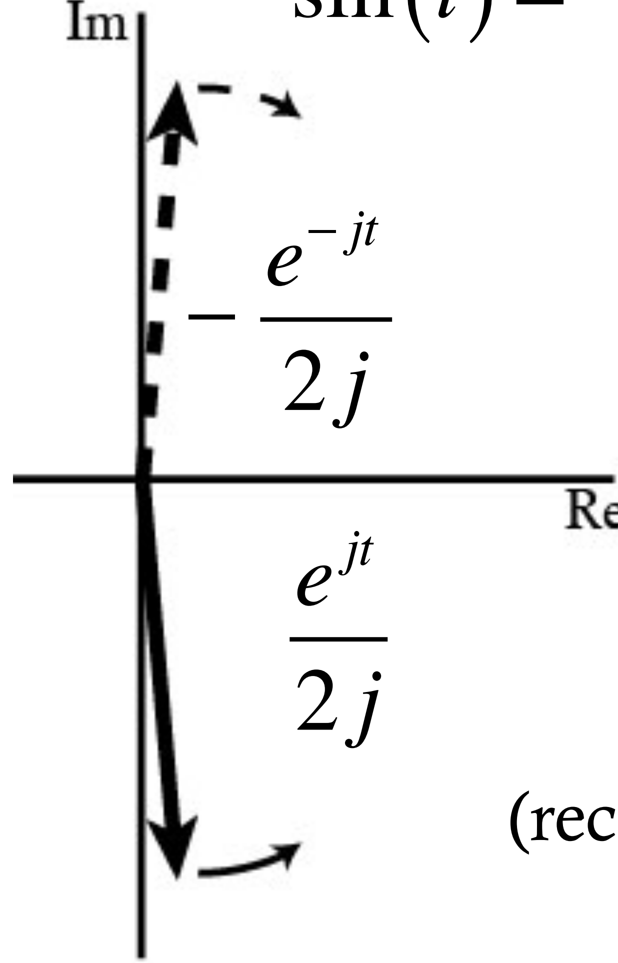
Sum of two phasors $y(t) = y_1(t) + y_2(t)$

Recall complex conjugate pairs of phasors

$$\cos(t) = \frac{e^{jt} + e^{-jt}}{2}$$



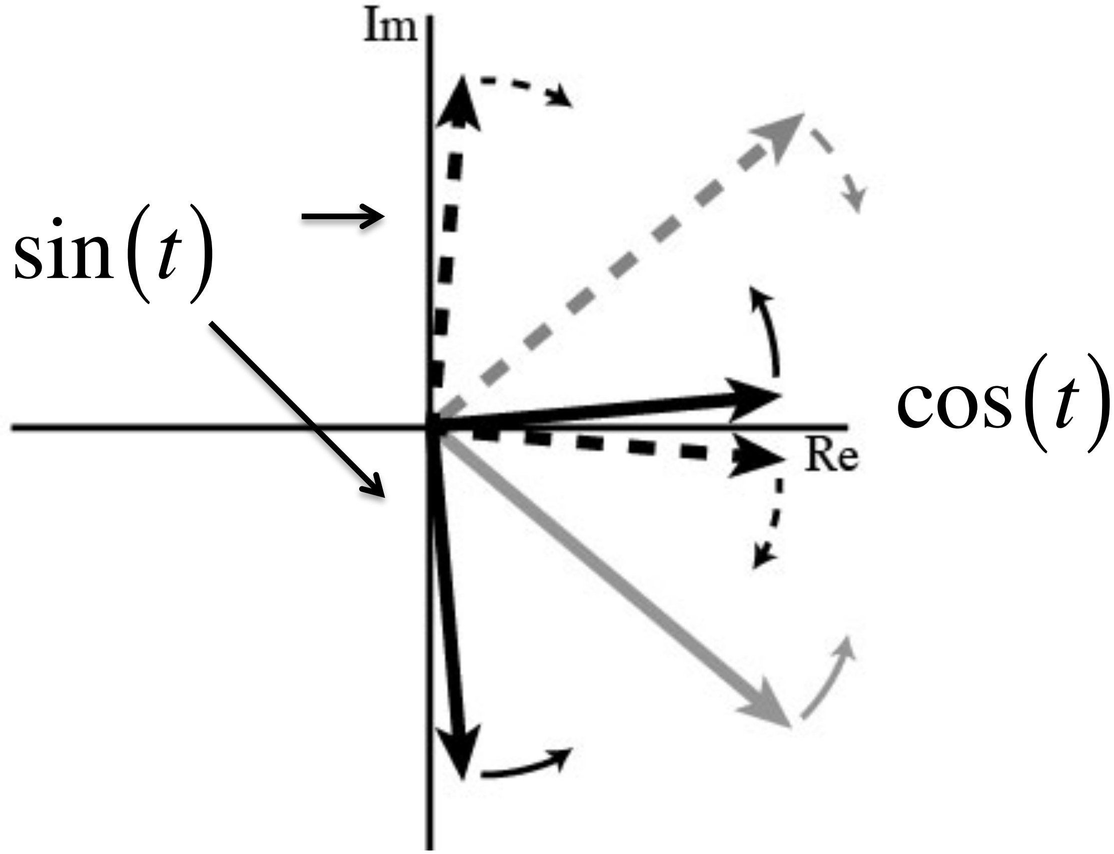
$$\sin(t) = \frac{e^{jt} - e^{-jt}}{2j}$$



(recall $\frac{1}{j} = -j$)

positive (solid) and negative (dashed) frequency.

Adding cos and sin conjugate pairs (black)…



…creates single conjugate pair (gray).

$$\cos(t) + \sin(t) = \underbrace{\sqrt{2}}_{\text{hypotenuse}} \cos\left(t - \frac{\pi}{4}\right)$$

Fourier Series

Applies only to *periodic* signals

Inverse Fourier Series

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jn\omega_0 t}$$

fundamental frequency ω_0

harmonic number

Fourier coefficient: stationary phasors (complex numbers) for each harmonic n determines magnitude and phase of that particular harmonic.

Any periodic signal $x(t)$ consists of a series of sinusoidal harmonics of a fundamental frequency ω_0.

For real $x(t)$, the phasor at each $n > 0$, spinning at $n\omega_0$ is paired with a complex conjugate phasor at $-n$, spinning in the other direction at $-n\omega_0$.

The "DC" harmonic, at $n = 0$, has a constant value of a_0.

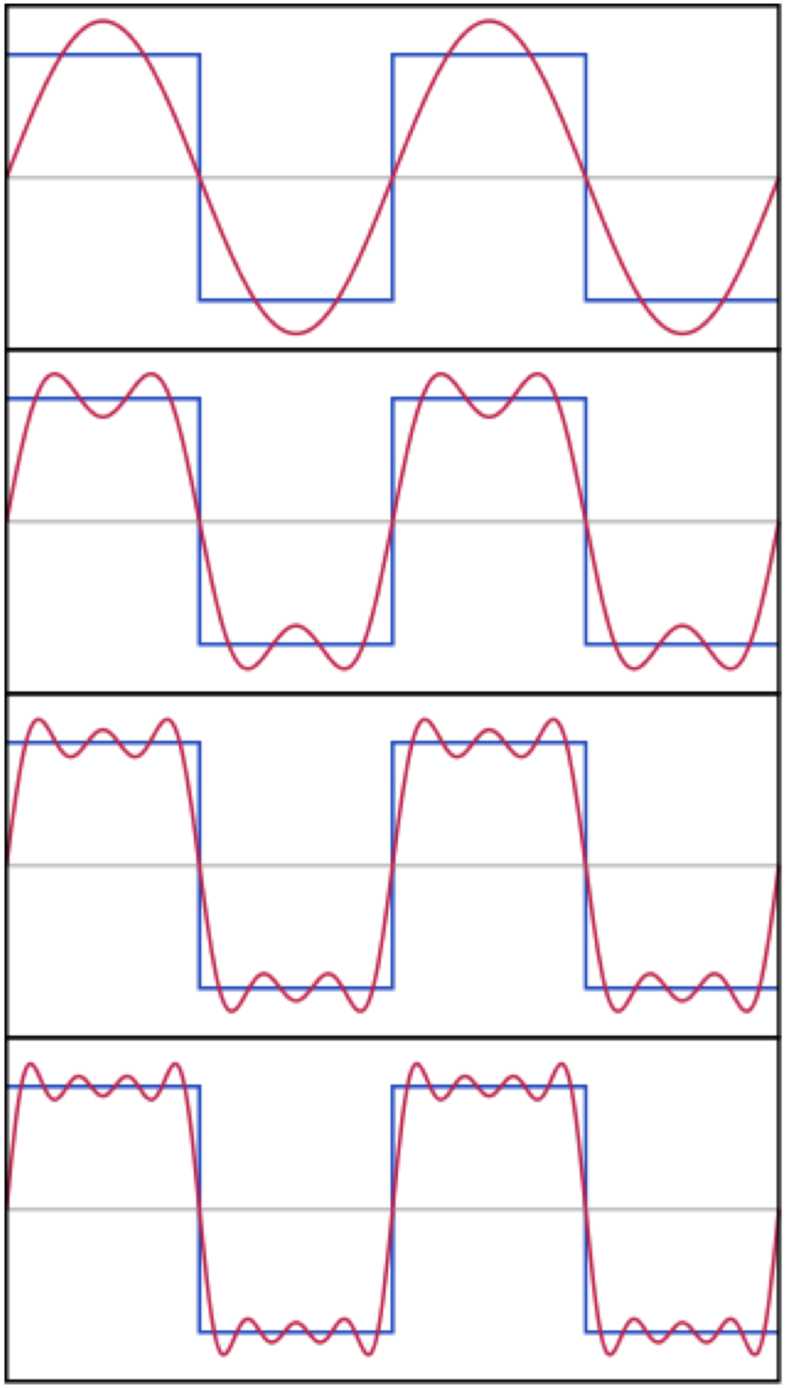
The n^{th} harmonic can also be written as a weighted sum of sin and cos at frequency $n\omega_0$.

$$A_n(\cos n\omega_0 t) + B_n(\sin n\omega_0 t)$$

creating a single sinusoid whose phase and amplitude are determined by real coefficients A_n and B_n.

The zero harmonic $n = 0$ (DC) is a cosine of zero frequency

$$A_{\mathrm{n}} \cos(0t)$$



Building a square wave by adding the odd harmonics: 1, 3, 5, 7…

An infinite number of harmonics are needed for a theoretical square wave.

The harmonics account for the harsher tone of the square wave (buzzer), compared to just the fundamental 1rst harmonic sinusoid (flute).

Fourier Series: How to find coefficient a_n

Inverse Fourier Series

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jn\omega_0 t}$$

Fourier Series

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} \, dt$$

stationary phasor for harmonic number n

period $T_0 = \dfrac{2\pi}{\omega_0}$

backwards-spinning phasor.

fundamental frequency

Periodic signal $x(t)$ consists of phasors forming the sinusoidal harmonics of ω_0.

Backward-spinning phasor $e^{-jn\omega_0 t}$ spins the entire set of phasors in $x(t)$, making the particular phasor $e^{jn\omega_0 t}$ stand still.

All other phasors complete n revolutions, integrating to 0.

Fourier Transform

Applies to *any* finite signal (not just periodic)

Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}\, d\omega$$

Fourier coefficient a_n has now become a continues function of frequency, $X(\omega)$, with phasors possible at *every* frequency.

Fourier Transform

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\, dt$$

As before, backwards-spinning phasor makes corresponding component of x(t) stand still.

$X(\omega)$ is a stationary phasor for any particular ω that determines the magnitude and phase of the corresponding phasor $e^{j\omega t}$ in $x(t)$.

The complex exponential $e^{j\omega t}$ forms an *orthogonal basis set* for any signal.

Each phasor passes through a linear system without affecting the system's response to any other.

To understand a linear system, all we need to know is what it does to $e^{j\omega t}$ for all values of ω.

This is the linear system's *frequency response.*

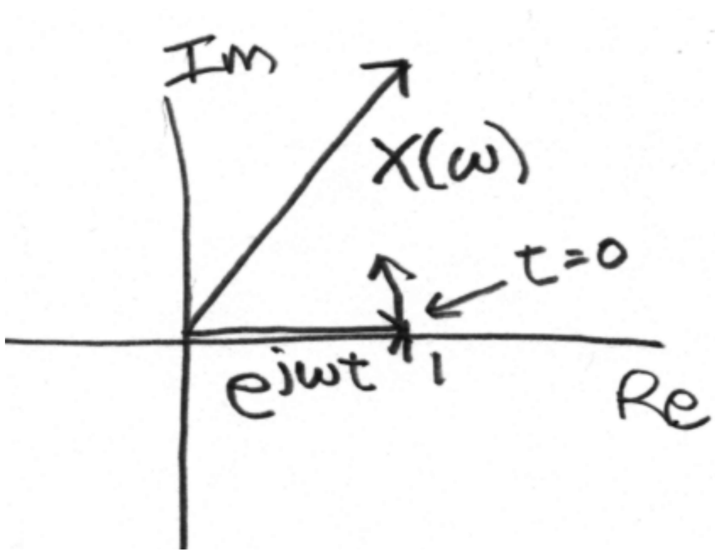
A linear system can only change the phase and amplitude of a given phasor, not its frequency, by multiplying it by a stationary phasor $H(\omega)$, the frequency response of the system.

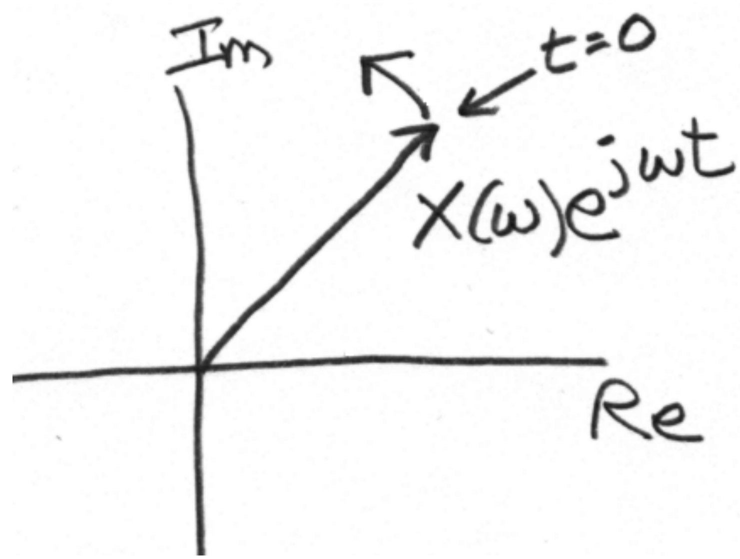
Frequency component $X(\omega)e^{j\omega t}$

The inverse Fourier Transform builds $x(t)$ from phasors at every frequency.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{X(\omega)e^{j\omega t}}_{} \, d\omega$$

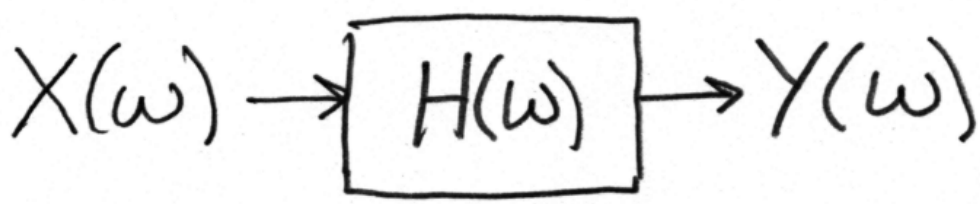
… stationary phasor $X(\omega)$ scales the magnitude and rotates the phase of unit spinning phasor $e^{j\omega t}$.





Systems modeled as Filters

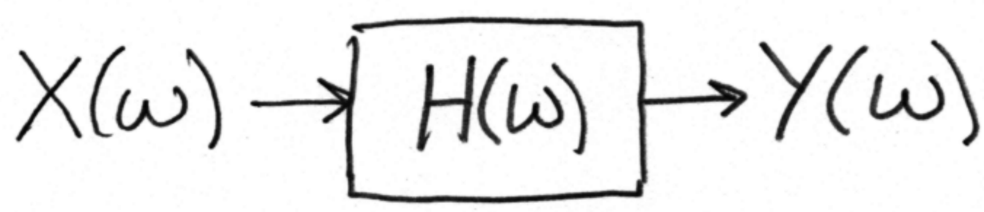
- We describe input and output signals as *spectra* $X(\omega)$ and $Y(\omega)$, the amplitude and phase of $e^{j\omega t}$ at ω.
- System's *transfer function* $H(\omega)$ changes the magnitude and phase of $X(\omega)$ to yield $Y(\omega)$ by multiplication.
- $H(\omega)$ is just another stationary phasor representing the amplitude *gain* and phase *shift* of the system.



$$Y(\omega) = H(\omega)\, X(\omega)$$

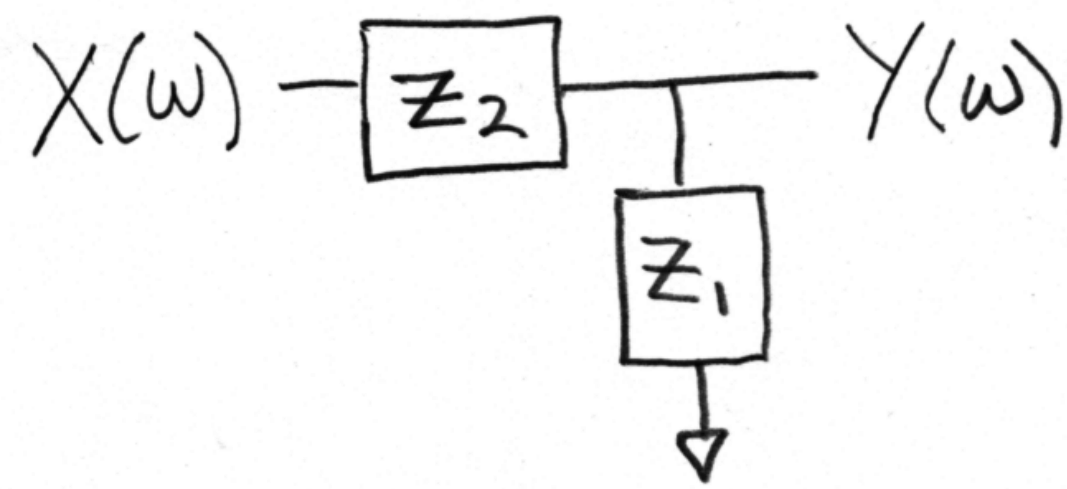
$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Systems modeled as Filters



$$H(\omega)=\frac{Y(\omega)}{X(\omega)}$$

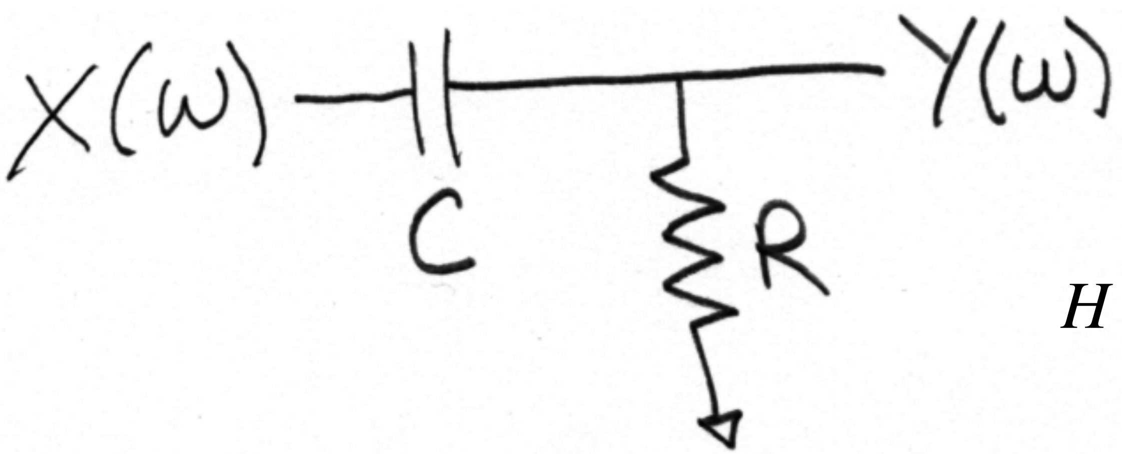
- Consider system with voltage divider of complex impedances.



$$H\left(\omega\right)=\frac{Z_1}{Z_1+Z_2}$$

- Same rule applies as with resistor voltage divider.
- Impedance divider changes the amplitude and phase of $X(\omega)e^{j\omega t}$.

Example: RC High-Pass Filter



$$H(\omega)=\frac{Y(\omega)}{X(\omega)}$$

$$H(\omega)=\frac{R}{R+\frac{1}{j\omega C}}=\frac{j\omega RC}{1+j\omega RC}$$

At high frequencies, acts like a piece of wire.

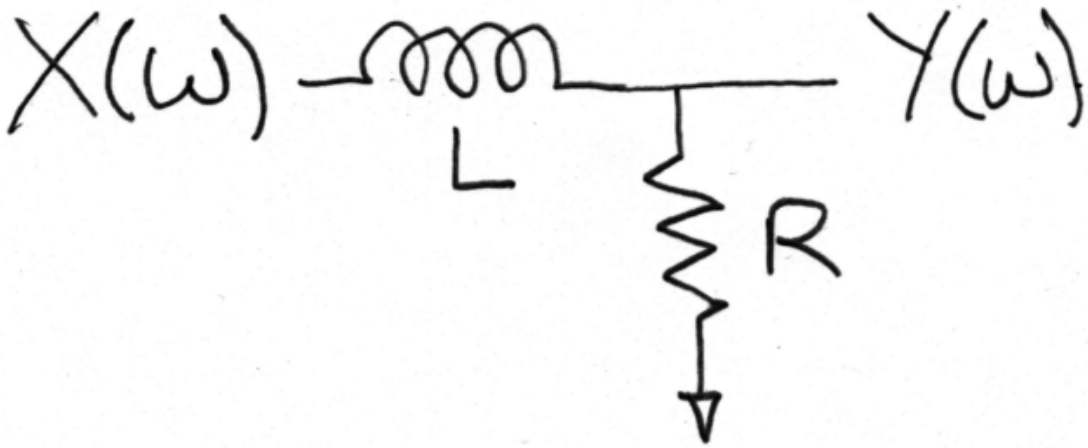
$$H(\omega)\cong 1,\quad \omega \gg \frac{1}{RC}$$

At low frequencies, attenuates and differentiates.

$$H(\omega)\cong j\omega RC,\quad \omega \ll \frac{1}{RC}$$

Key frequency is reciprocal of time constant *RC*.

Example: LR Low-Pass Filter



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$H(\omega) = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R}}$$

At low frequencies, acts like a piece of wire.

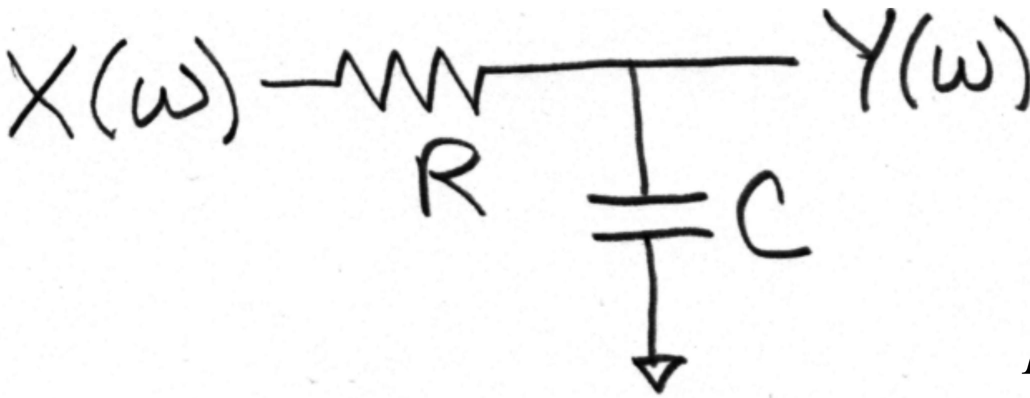
$$H(\omega) \cong 1, \quad \omega << \frac{R}{L}$$

At high frequencies, attenuates and integrates.

$$H(\omega) \cong \frac{R}{j\omega L}, \quad \omega >> \frac{R}{L}$$

Key frequency is reciprocal of time constant L/R.

Example: RC Low-Pass Filter



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

At low frequencies, acts like a piece of wire.
(*assuming no current at output*)

$$H(\omega) \cong 1, \quad \omega << \frac{1}{RC}$$

At high frequencies, attenuates and integrates.

$$H(\omega) \cong \frac{1}{j\omega RC}, \quad \omega >> \frac{1}{RC}$$

Key frequency is reciprocal of time constant *RC*.

Decibels – ratio of gain (attenuation)

- 1 Bell = 10 dB = order of magnitude in power

$$1\ \text{dB} \equiv 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right)$$

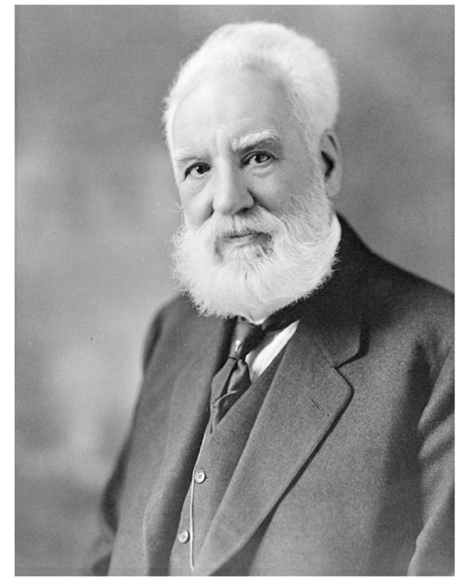
so if $P_{\text{in}} = 1$ W and $P_{\text{out}} = 100$ W ➔ 20 dB

- Since

$$\text{power} \propto \text{voltage}^2$$

$$1\ \text{dB} \equiv 20\log_{10}\left(\frac{V_{out}}{V_{in}}\right)$$

so if $V_{\text{in}} = 1$ V and $V_{\text{out}} = 10$ V ➔ 20 dB

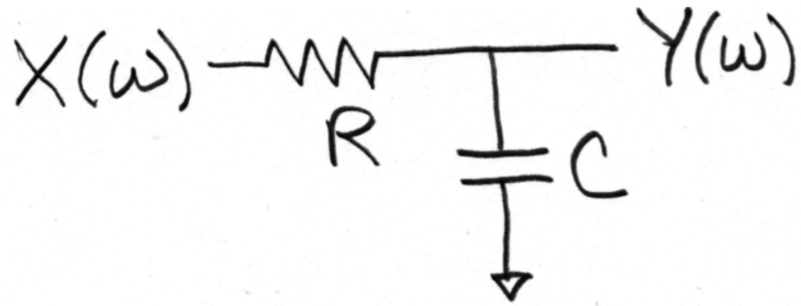


Alexander Graham Bell

- dB is a pure ratio (no units) as opposed to dB_m (power compared to 1 mW), dB_V (voltage compared to 1 V), dB_SPL (sound pressure level compared to threshold of hearing), etc.

Magnitude and Phase of Low-Pass Filter

Recall low-pass filter:



$$H(\omega)=\frac{1}{1+j\omega RC}$$

At *corner* (or *"cut-off"*) *frequency*, $\omega_C = 1/RC$,

$$H(\omega)=\frac{1}{1+j}\cdot\frac{1-j}{1-j}=\frac{1-j}{2}$$

<u>Magnitude</u> (Gain/Attenuation)

$$\left|H(\omega)\right|=\left|\frac{1-j}{2}\right|=\frac{1}{\sqrt{2}}$$

$$\left|H(\omega)\right|\cong -3\text{dB}$$

<u>Phase</u>

$$\angle H(\omega)=\arctan\left(\frac{-1/2}{1/2}\right)$$

$$\angle H(\omega)=-45^\circ$$

"Bode" Plot of Low Pass Filter (previous slide)

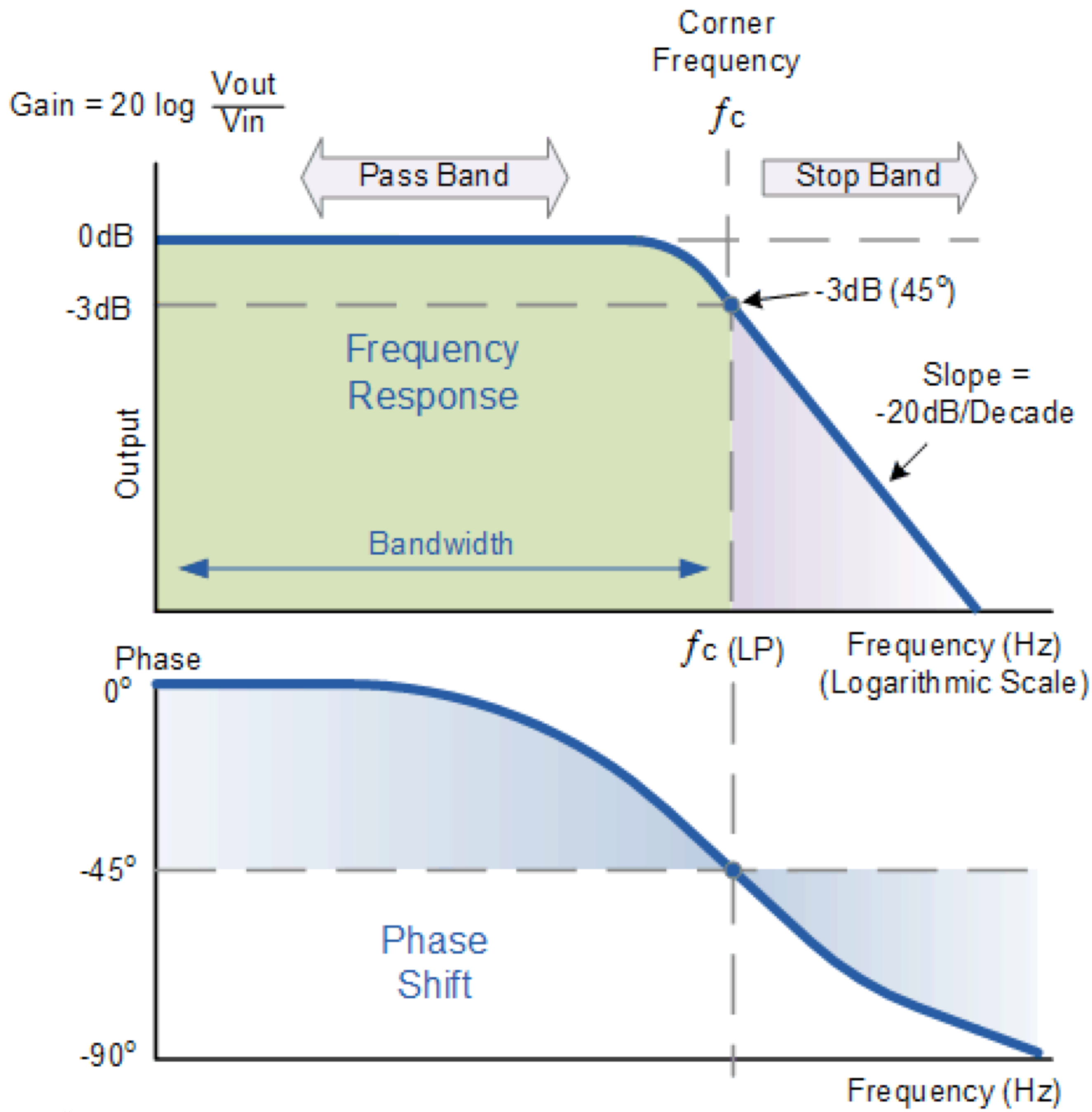
Simply a log/log plot of

$$\left|H(\omega)\right|$$

and

$$\angle H(\omega)$$

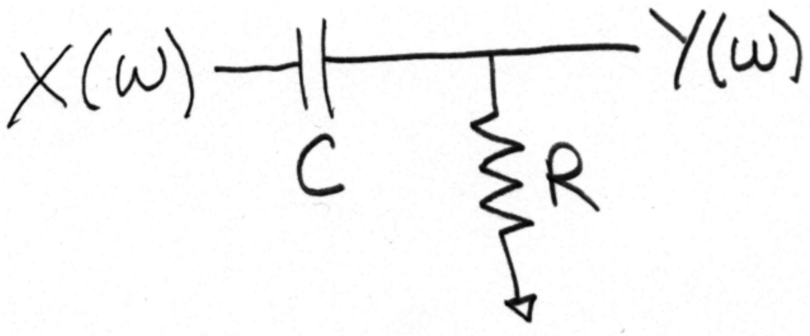
(this one vs. f, not ω)



http://www.electronics-tutorials.ws/filter/filter_2.html

Magnitude and Phase of High-Pass Filter

Recall high-pass filter:



$$H(\omega)=\frac{j\omega RC}{1+j\omega RC}$$

At *cut-off frequency*, $\omega_C = 1/RC$,

$$H(\omega)=\frac{j}{1+j}\cdot\frac{1-j}{1-j}=\frac{1+j}{2}$$

<u>Magnitude</u>

$$\left|H(\omega)\right|=\left|\frac{1+j}{2}\right|=\frac{1}{\sqrt{2}}$$

$$\left|H(\omega)\right|\cong -3\text{dB}$$

<u>Phase</u>

$$\angle H(\omega)=\arctan\left(\frac{1/2}{1/2}\right)$$

$$\angle H(\omega)=45^\circ$$

"Bode" Plot of High Pass Filter (previous slide)

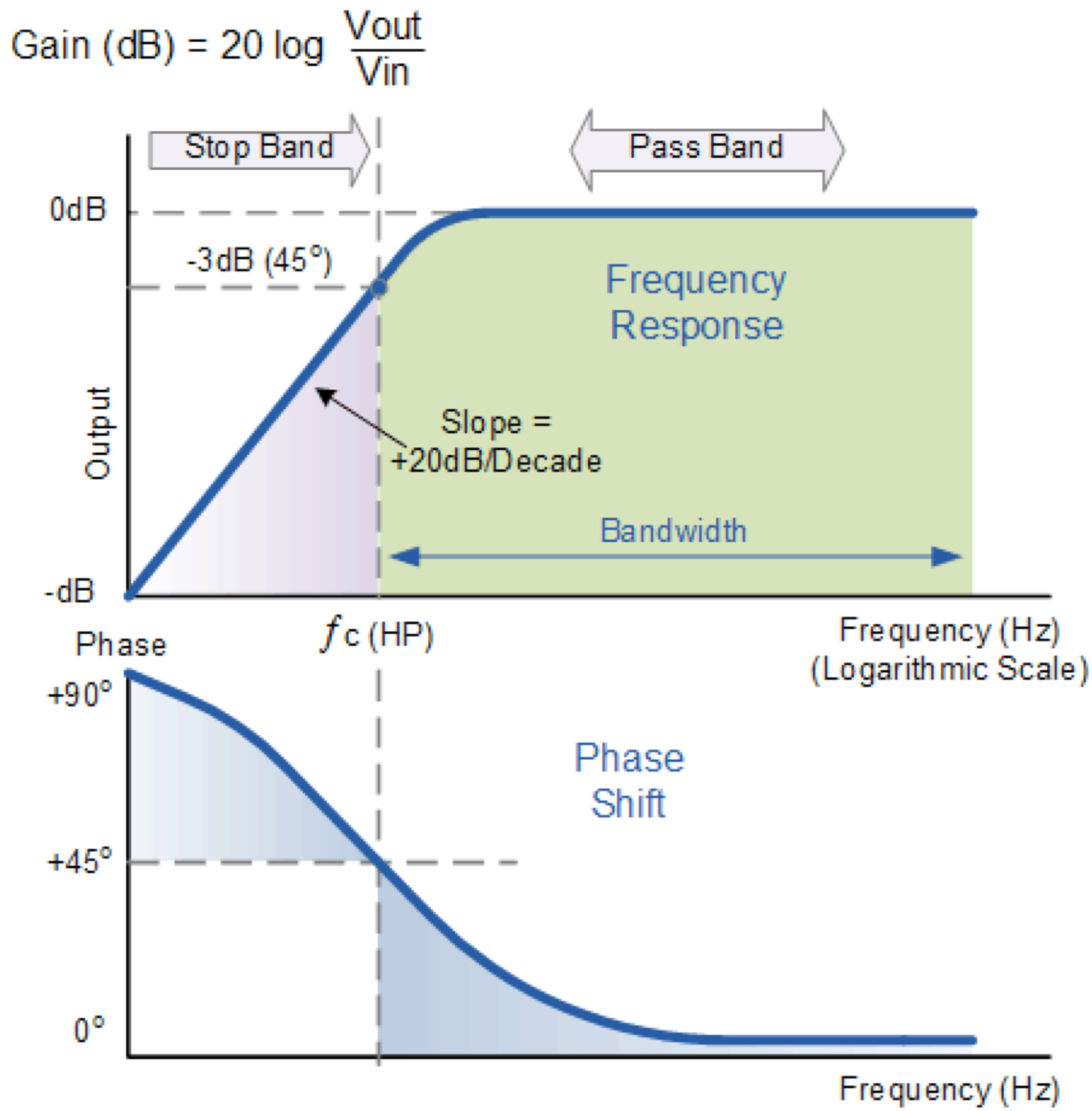
Simply a log/log plot of

$$\left|H(\omega)\right|$$

and

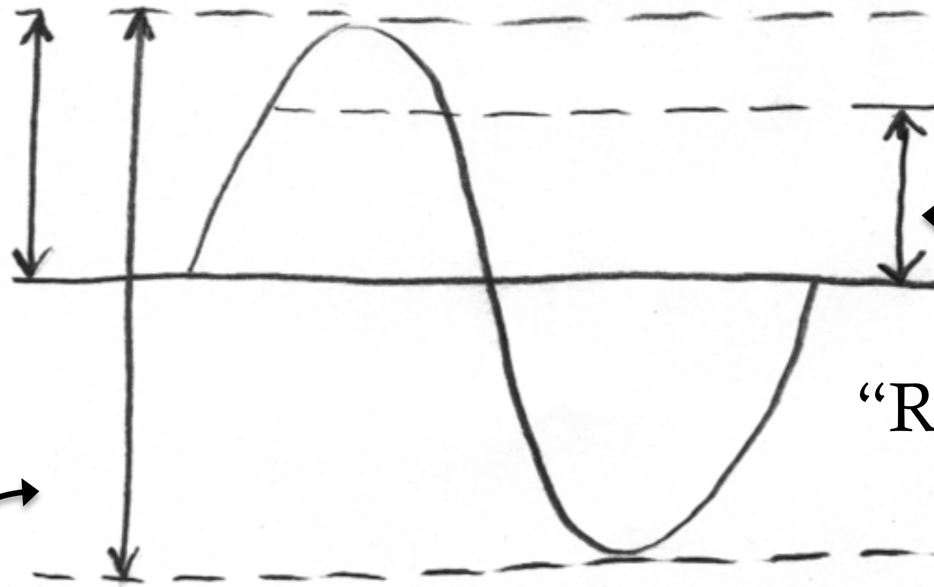
$$\angle H(\omega)$$

(this one vs. f, not ω)



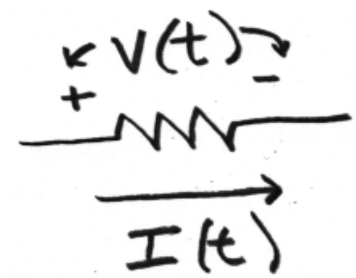
http://www.electronics-tutorials.ws/filter/filter_2.html

Values for AC Voltage



- Any sinusoidal signal $V(t)$ has all three values.
- Since $\sin^2 + \cos^2 = 1$, and since $\sin^2$ and $\cos^2$ must each have the same *mean* value, each must have a mean value of ½.
- Or put another way: $$\cos^2(\omega t) = \frac{1+\cos(2\omega t)}{2}$$ mean = ½
- Therefore, for a sinusoid $V_{RMS} = \dfrac{V_P}{\sqrt{2}}$

RMS used to compute AC Power in Resistor



When V and I are in-phase (resistor), average power is defined as in DC.

Energy is not stored in the resistor, but simply dissipated as heat.

For any signal in a resistor:

$$P = V_{RMS} \times I_{RMS} = \frac{\left(V_{RMS}\right)^2}{R} = \left(I_{RMS}\right)^2 R$$

Power in a resistor may be computed from V_P or I_P for sinusoids, or from V_{RMS} or I_{RMS} for any signal.

For sinusoids:

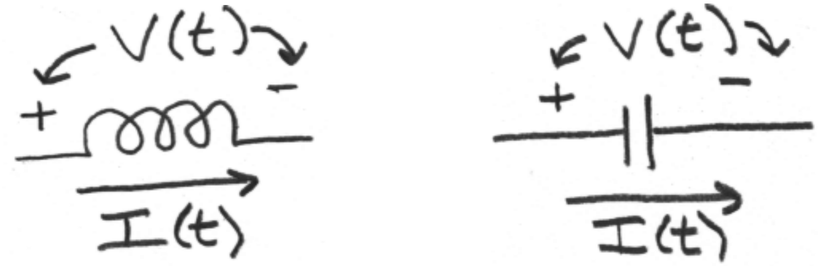
$$P = \frac{1}{2} V_P \times I_P = \frac{1}{2} \frac{\left(V_P\right)^2}{R} = \frac{1}{2}\left(I_P\right)^2 R$$

because

$$V_{RMS} = \frac{V_P}{\sqrt{2}} \quad \text{and} \quad I_{RMS} = \frac{I_P}{\sqrt{2}}$$

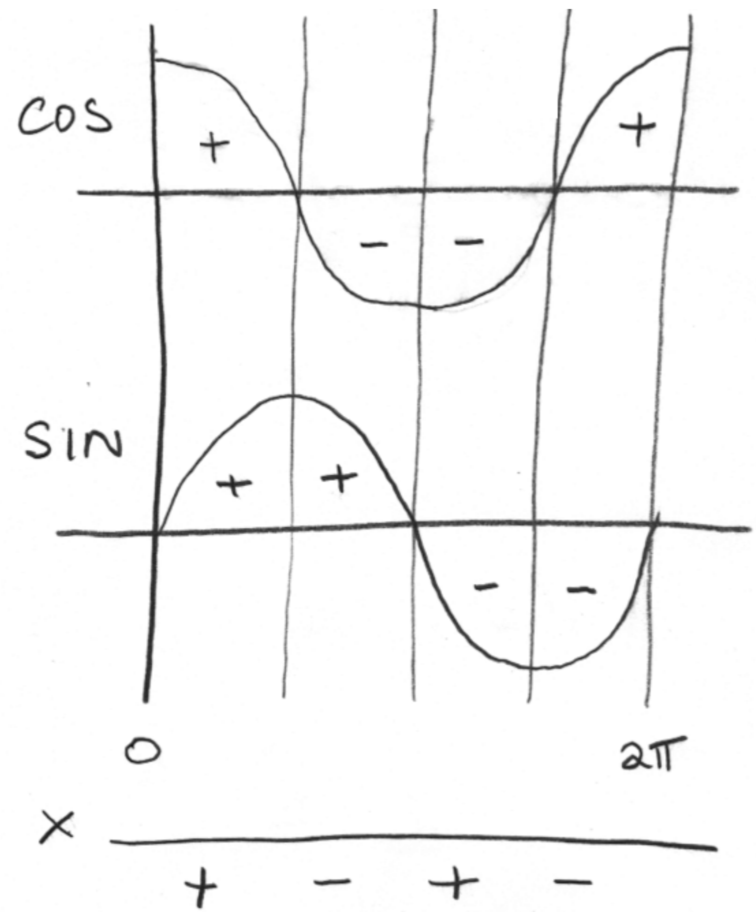
AC Power in Capacitor or Inductor

Since V_{RMS} and I_{RMS} are 90° out-of-phase in capacitor or inductor, the power dissipated is 0.



$$\cos(\omega t)\sin(\omega t) = \frac{\sin(2\omega t)}{2}$$

average = 0



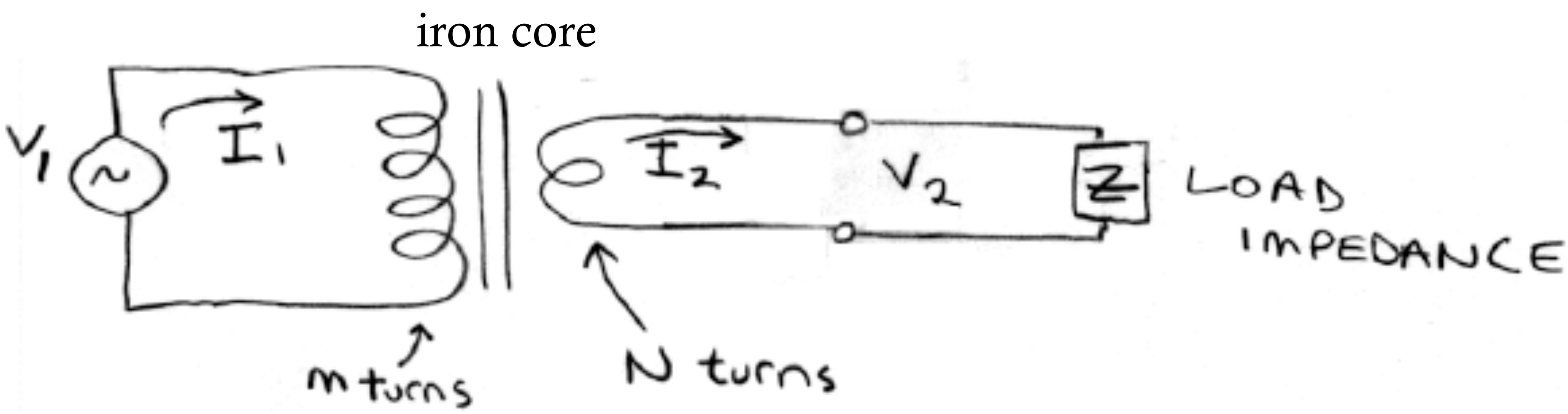
sin and cos have *zero correlation*:

The integral of their product = 0

Thus no heat is dissipated, all stored energy returned to circuit

sin and cos have 0 correlation

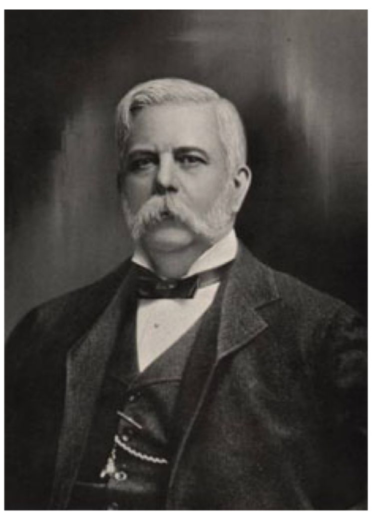
Transformer



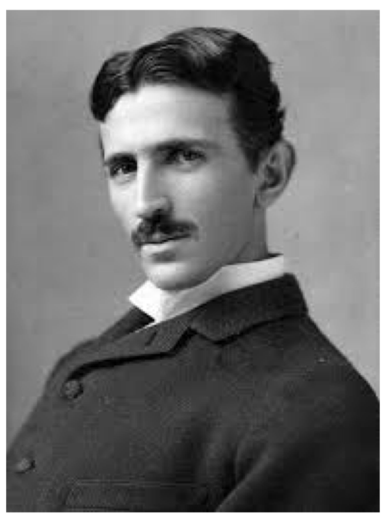
- Allows voltage (AC) to be changed: $V_2 = \frac{N}{M} V_1$
- Extremely efficient at preserving power: $V_1 I_1 \cong V_2 I_2$
- Voltages and currents in RMS assumed to be sinusoids
- Can be *step-up* transformers (N>M) or *step-down* (N<M)
- Permits efficient high-voltage power transmission, with small current: thus little I^2R energy wasted in long wires.
- Transformers also used to provide isolation for safety.

World's Fair
Chicago
1893
Tesla and
Westinghouse
(AC) beat
Edison (DC).

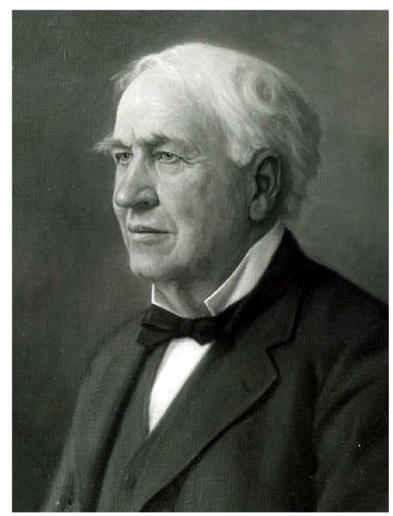
George
Westinghouse



Nikola
Tesla



Thomas
Edison

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GENERAL ELECTRIC COMPANY.
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CHICAGO
44 BROAD ST
NEW YORK
620 ATLANTIC AVE
BOSTON

High Voltage DC power lines

- DC recently making a comeback.
- New efficient systems for converting between DC and AC.
- Especially good for long distances with renewable sources such as solar and wind.
- Easier because power grids don't need to be synchronized with each other.
- More efficient transmission (no radiation)
- Narrower rights-of-way (no radiation)

Summary of AC

- Introduces 2 new linear components: inductor and capacitor, that perform integration and differentiation of voltage and current.
- AC signals are composed of sinusoids, which are formed from pairs of phasors.
- Linear differential equations can be solved by algebra using complex impedance.
- Frequency response of a system $H(\omega)$ relates spectra of output signal to input signal.
- Linear systems change only amplitude and phase, but never frequency.