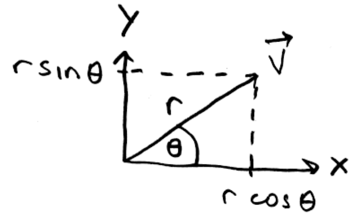
Review of Sine & Cosine

$\cos\theta = \frac{x}{r}$



$\sin\theta = \frac{y}{r}$

saying $\cos^2\theta + \sin^2\theta = 1$ is just the pythagorean theorem

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$x^2 + y^2 = r^2$$

83

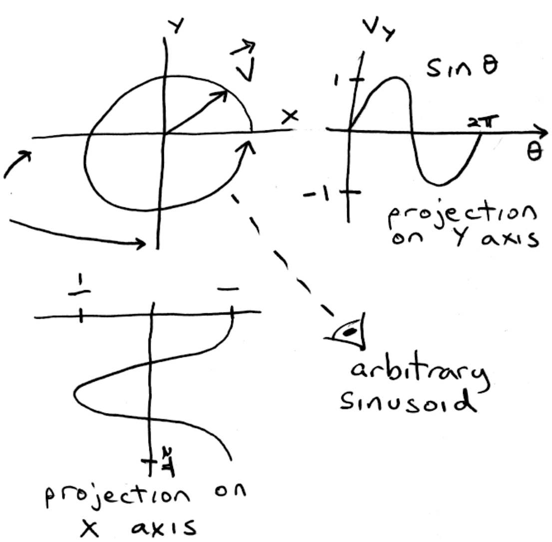
Review of Sine & Cosine

when $r = 1$

$V_x = \cos\theta$

$V_y = \sin\theta$

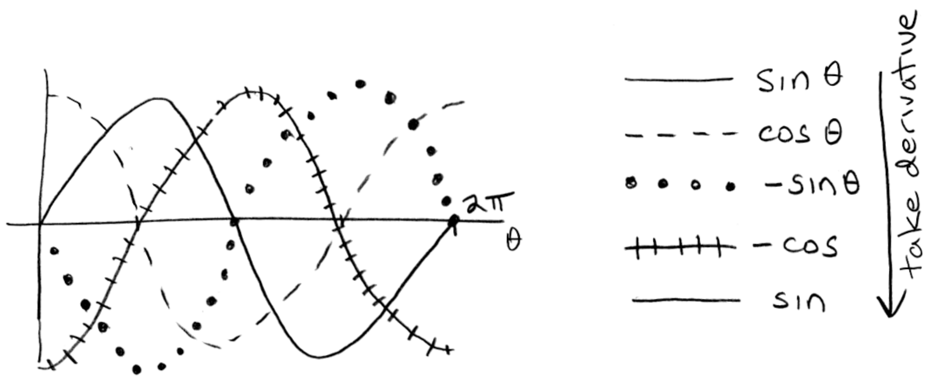
cardinal axes are just an arbitrary choice



Sin vs cos vs any sinusoid is just a matter of where you say $\theta = 0$

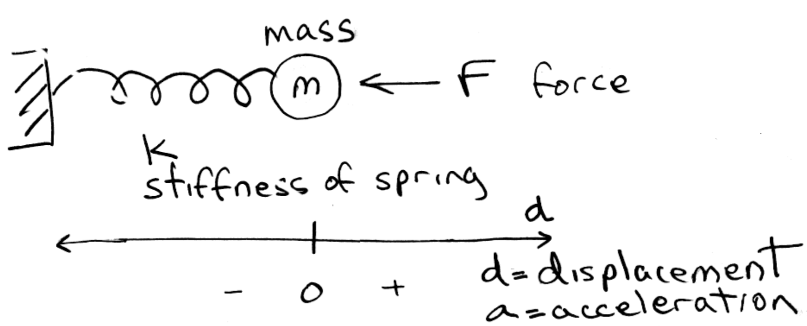
84

Derivative shifts 90° to the left



Taking a second derivative inverts a sinusoid.

Hooke's Law

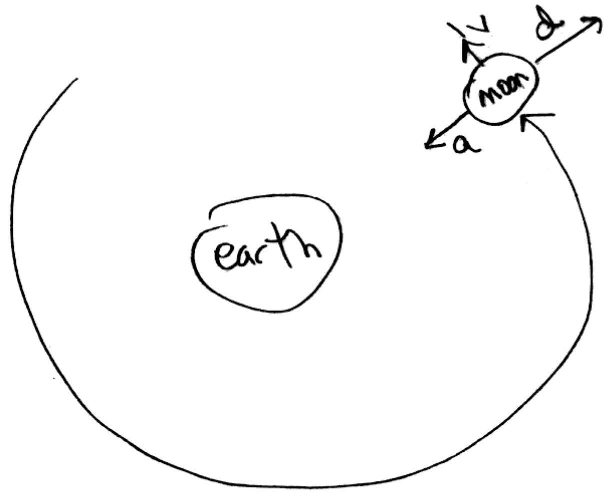


$$F = MA$$
$$F = -kd$$
$$\Rightarrow d = -\left(\frac{M}{k}\right)a$$

constant

Sinusoids result when a function is proportional to its own negative second derivative.

Pervasive in nature

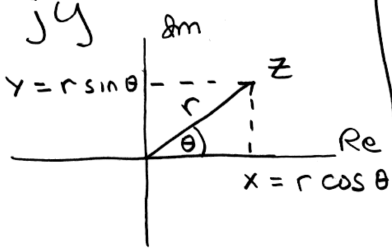


also swings, flutes, electron orbits, light waves, sound waves…

Complex numbers

- Cartesian and Polar forms on complex plane.
- Not vectors, though they add like vectors.
- Can multiply two together (not so with vectors).

$z = x + jy$



$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

we say that $-\pi < \theta \leq +\pi$ to make it unique, though it θ actually periodic

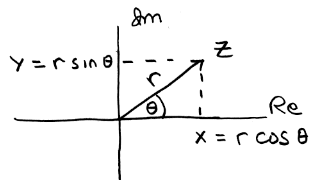
$\theta + k2\pi$

$k = 0, \pm 1, \pm 2, \ldots$

http://en.wikipedia.org/wiki/Phasor for nice animations

Complex Numbers

- How to find *r*



$z = x + jy$

"modulus" of z, → $|z| = \sqrt{x^2 + y^2}$

"absolute value" is just a special case where $y = 0$.

$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

$= r\sqrt{\underbrace{\cos^2\theta + \sin^2\theta}_{1}}$

$= r$, which is always ≥ 0

Note: this is not $\sqrt{z^2}$, but rather the length of the line, r

89

Complex Exponentials – Phasor

- All algebraic operations work with complex numbers.
- What does it mean to raise something to an imaginary power?

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \cdots$$

$$\sin(t) = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} \cdots$$

$$\cos(t) = 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 \cdots$$

$$j\sin(t) = 0 + jt + 0 - j\frac{t^3}{3!} + 0 + j\frac{t^5}{5!} \cdots$$

$$e^{jt} = 1 + jt - \frac{t^2}{2} - j\frac{t^3}{3!} + \frac{t^4}{4!} + j\frac{t^5}{5!} \cdots$$

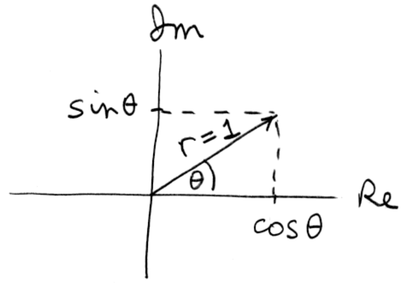
$$\boxed{e^{jt} = \cos(t) + j\sin(t)}$$

Euler's Identity

90

Cartesian and Polar forms

First,
Fixed <u>unit</u> phasor, $r = 1$



$$e^{j\theta} = \cos\theta + j\sin\theta$$

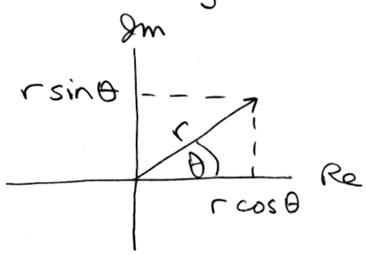
Euler's Identity

$r = |e^{j\theta}| = 1$ because $\sin^2\theta + \cos^2\theta = 1$

91

Cartesian and Polar forms (cont…)

Now, for any complex number $z = x + jy$



multiply Euler's Identity by r

$$\underbrace{r e^{j\theta}}_{\text{polar: } r,\,\theta} = \underbrace{\underbrace{r\cos\theta}_{x} + j\,\underbrace{r\sin\theta}_{y}}_{\text{cartesian: } x,\,y}$$

92

Multiplying by a Complex Number

$$z = x + jy = re^{j\theta}$$

rotates by θ and scales by r

messy in cartesian coordinates $\rightarrow$ $(x_1 + jy_1)(x_2 + jy_2) =$

$$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) =$$

$$\underbrace{r_1 r_2}_{\text{scale each other}} e^{j\overbrace{(\theta_1 + \theta_2)}^{\text{rotate each other}}}$$

93

Dividing by a Complex Number

How to eliminate complex denominator

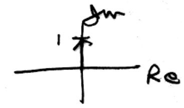
Multiply denominator by complex conjugate

$$\frac{1}{a+jb} = \frac{1}{a+jb}\,\frac{a-jb}{a-jb} = \underbrace{\frac{a}{a^2+b^2}}_{\text{real part}} - \underbrace{\frac{jb}{a^2+b^2}}_{\text{imaginary part}}$$

94

Rotating by + or - 90°

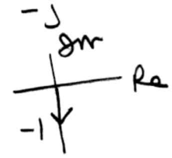
$$j = \underset{\uparrow r}{1} e^{j\overset{\theta}{\left(\frac{\pi}{2}\right)}}$$



Multiplying by j rotates any complex number by 90°

dividing by j rotates by −90°

because $\frac{1}{j} = \frac{1}{j}\,\frac{-j}{-j} = -j = 1e^{-j\frac{\pi}{2}}$



Examples

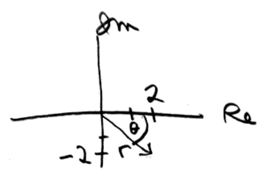
convert the following complex numbers to polar coordinates $re^{j\theta}$

$r \geq 0 \quad -\pi < \theta \leq \pi$

① $\boxed{2 - 2j}$

$x = 2$

$y = -2$



$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

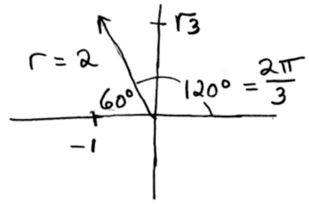
$\theta = -45° = -\frac{\pi}{4}$

therefore, $2 - 2j = \boxed{2\sqrt{2}\, e^{-j\frac{\pi}{4}}}$

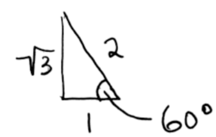
(2) $\boxed{-1+\sqrt{3}\,j}$

$x=-1$

$y=\sqrt{3}$



$\boxed{2e^{j\frac{2\pi}{3}}}$



in general

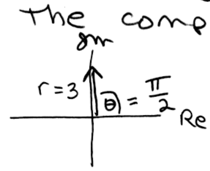
$$re^{j\theta} = \sqrt{x^2+y^2}\; e^{j\arctan\left(\frac{y}{x}\right)}$$

97

Going the other way

convert the following complex numbers to cartesian coordinates $x+jy$ drawing a picture in the complex plane

(1) $3e^{j\frac{\pi}{2}}$



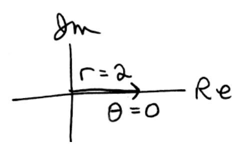
$x=0$, $y=3$

$z = 0 + j3$

(2) $-2e^{-j3\pi} = \underbrace{-2e^{-j\pi}}_{\substack{\text{since} \\ e^{j\theta}=e^{j(\theta+2\pi K)} \\ K=0,\pm1,\pm2,\ldots}} = \underbrace{2e^{0}}_{\substack{\text{since} \\ e^{j\theta}=-e^{j(\theta\pm\pi)}}} = 2+j0$

$x=2$

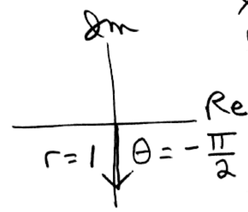
$y=0$



98

③ $je^{j\pi} = e^{j\frac{\pi}{2}}e^{j\pi} = e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}} = 0 - j$

$x = 0$
$y = -1$



In general
$x = \mathrm{Re}\{re^{j\theta}\}$ $y = \mathrm{Im}\{re^{j\theta}\}$

The "squiggly" bracket: not an algebraic expression.

y itself is real: the coordinate on the imaginary axis

99

Let's review the dimensionality of phase and frequency...

θ = phase = angle

usually in radians, but can be degrees, or cycles

1 cycle = 360 degrees = 2π radians

$e^{j(\,)}$ this is phase

100

phase = frequency × time, as in

$$e^{j(\omega t)} = e^{j(2\pi f t)}$$

$$\omega = 2\pi f$$

frequency in radians/sec

frequency in cycles/sec

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

period, seconds/cycle

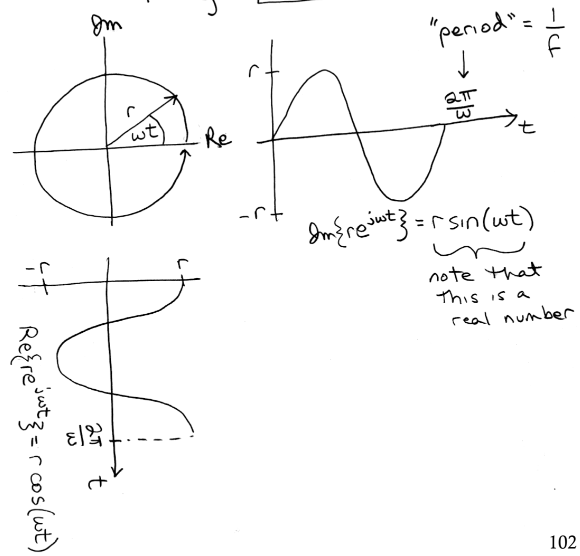
101

$$\underset{\text{amplitude}}{r}\, e^{\overset{\text{frequency}}{j\omega t}} = r\cos(\omega t) + j\, r\sin(\omega t)$$

$\boxed{\text{spinning arrow in the complex plane}}$

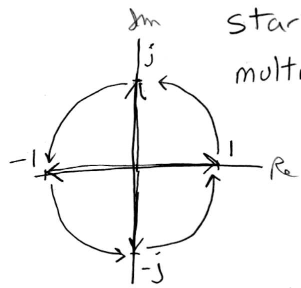
Now make the phasor spin at

$\omega = 2\pi f$



102

Multiplying by j shifts the phase of the spinning phasor by 90°



start with 1

multiply by j, you get $j, -1, -j, 1$

shift by $+90° = +\frac{\pi}{2}$

phasors do the same thing when you take their derivative:

$$\frac{de^{jt}}{dt} = j e^{jt} \qquad (\omega = 1)$$

$$\frac{d^2 e^{jt}}{dt^2} = j \cdot j\, e^{jt} = -e^{jt}$$

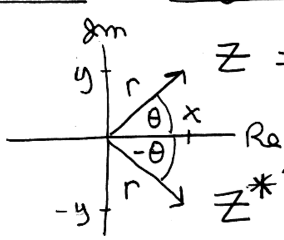
$$j = e^{j\frac{\pi}{2}}$$

$\theta = 90°$

Just like a sinusoid and Hooke's Law: shifts 90° with each derivative

Complex Conjugates

Complex conjugates – reflect across x-axis



$$Z = x + jy = r e^{j\theta}$$

asterix means complex conjugate

$$Z^* = x - jy = r e^{-j\theta}$$ negative sign

product

$$(x + jy)(x - jy) = x^2 + y^2 = r^2$$

or with phasors, phase cancels out

$$(r e^{j\theta})(r e^{-j\theta}) = r^2 e^{j(\theta - \theta)} = r^2 e^0 = r^2$$

$$Z\, Z^* = |Z|^2$$ "modulus"

Voltages and Currents are Real

getting Re{z} and Im{z} using algebra

$$\mathrm{Re}\{z\} = x = \frac{(x+jy)+(x-jy)}{2} = \frac{z+z^*}{2}$$

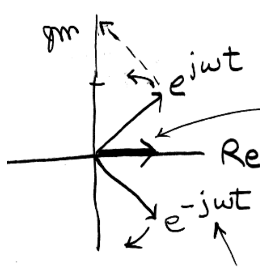
$$\mathrm{Im}\{z\} = y = \frac{(x+jy)-(x-jy)}{2j} = \frac{z-z^*}{2j}$$

The imaginary coordinate y is itself real.

$z = x + jy$ ← this is imaginary.

105

Cosine is sum of 2 phasors



complex conjugates

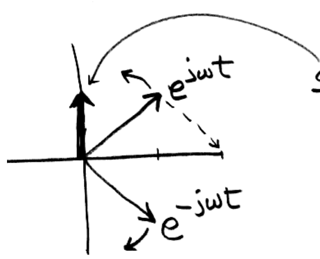
$$\cos(\omega t) = \mathrm{Re}\{e^{j\omega t}\} = \frac{e^{j\omega t}+e^{-j\omega t}}{2}$$

to get cos: subtract off imaginary part

negative frequency, backwards spinning phasor, complex conjugate

106

Sine is difference between 2 phasors



$$\sin(\omega t) = \underbrace{\Im m\{e^{j\omega t}\}}_{} = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

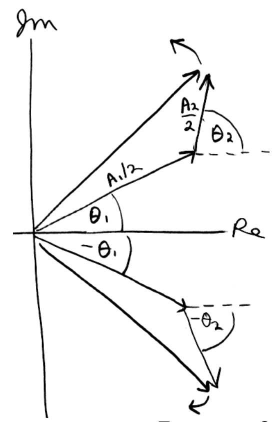
this is a real number so must divide by j

$$\boxed{\text{to get sin: subtract off real part.}}$$

107

Superposition Revealed

The forward and backward spinning phasors within two sinusoids at the same frequency group independently.



Rotate as 2 rigid bodies

$$A_1\cos(\omega t + \theta_1) + A_2\cos(\omega t + \theta_2)$$

<u>is</u> a sinusoid of frequency ω

$$\underbrace{\left[\frac{A_1}{2}e^{j\theta_1} + \frac{A_2}{2}e^{j\theta_2}\right]}_{} e^{j\omega t} + \underbrace{\left[\frac{A_1}{2}e^{-j\theta_1} + \frac{A_2}{2}e^{-j\theta_2}\right]}_{} e^{-j\omega t}$$

Single complex number and its complement completely describe its phase and amplitude.

108